Economic Theory (2021) 72:615–639
https://doi.org/10.1007/s00199-020-01301-0

RESEARCH ARTICLE

Eventual periodicity in the two-sector RSL model: equilibrium vis-à-vis optimum growth

Liuchun Deng[1] · Minako Fujio[2] · M. Ali Khan[3]

Received: 5 August 2019 / Accepted: 19 July 2020 / Published online: 18 August 2020

Abstract

This paper investigates the well-known phenomenon of *eventual* periodicity of Li–Yorke chaos in the context of the two-sector Robinson–Shinkai–Leontief model of economic growth. It locates its (i) presence under specific parameter restrictions that include the extreme classical saving specification, and its (ii) absence in savings generated by the optimization of an infinitely-lived representative agent with perfect foresight. These results in which rare events, chaos and stability are all brought together under the rubric of upward and downward inertia, while of substantive economic interest of their own, also highlight phenomena in economic dynamics that may go towards a clearer definitional understanding of chaotic systems.

Keywords Two-sector model · Equilibrium growth · Optimum growth · Eventual periodicity · Chaos

JEL Classification C60 · D90 · O21

This paper is dedicated to the memory of T. N. Srinivasan, a pioneer of two-sector growth theory, a past editor of *Econometrica,* and also a mentor and role-model to so many others going well beyond the authors of this modest contribution, modest especially when keeping TN's high standards in mind. Part of the results reported in this essay were presented at the 26th Annual Symposium of the *Society for Nonlinear Dynamics and Econometrics* in Keio University on March 19, 2018. The authors gratefully acknowledge stimulating conversation and exchanges with Bob Barbera, Jess Benhabib, Chris Carroll, Edmund Crawley, Tapan Mitra, Kazuo Nishimura, and Kevin Reffett, and finally, the encouragement of the Associate Editor of this journal and the careful reading of his/her two anonymous referees. Liuchun Deng acknowledges the support of the Start-up Grant from Yale-NUS College.

✉ Liuchun Deng
liuchun.deng@yale-nus.edu.sg

1 Division of Social Sciences, Yale-NUS College, Singapore, Singapore

2 Department of Economics, Yokohama National University, Yokohama, Japan

3 Department of Economics, The Johns Hopkins University, Baltimore, MD, USA

But this long run is a misleading guide to current affairs. In the long run we are all dead. Economists set themselves too easy, too useless a task if in tempestuous seasons they can only tell us that when the storm is long past the ocean is flat again.[1]

Keynes (1923)

1 Introduction

Since Li–Yorke's discovery that period-three implies chaos, it has been long known that the specific formulation of chaos that they defined and studied could be unobservable; and thus, a chaotic map may possibly lead to as smooth and as non-erratic a dynamic as one might imagine; see Li and Yorke (1975). In the ensuing discussion, Nathanson (1977) constructed a piecewise-linear and Li–Yorke chaotic map on an interval to itself that did lead to *eventual* periodicity of any order; that is, to trajectories that converge in finite periods to a period-p cycle, $p > 2$ a given integer, starting from almost any initial point.[2] This striking result shows that the Li–Yorke chaos could not only be non-observable but also extremely well-behaved. The question is whether the result is of any consequence for models in economic dynamics and goes beyond dynamical systems based on the unit interval.

In this paper, we exploit the recent extension of Nathanson's construction due to Khan and Rajan (2017) to elaborate the concept of eventual periodicity in the two-sector Robinson–Shinkai–Leontief (RSL) model of economic growth.[3] This, our primary contribution, is to present an economic model that exhibits such a seemingly paradoxical feature of the dynamics, and thereby provides a clearer definitional understanding of chaos. Perhaps somewhat more surprisingly, being a benchmark model with linear technology, the equilibrium dynamics of the RSL model has never been satisfactorily established.[4] This charting is perhaps the second contribution of this paper. In search of eventual periodicity, we provide a comprehensive characterization, one that goes substantially beyond the existing work that predominantly focuses on the steady-state analysis,[5] and thereby connects to Keynes' well-known statement being used as the epigraph to this work. In presenting a growth model almost all of whose

[1] Keynes (1923, p. 80); also in Keynes' *Collected Works* Volume IV, p. 65. Mann (2017, p. 51) uses these three sentences as the epigraph to his book, and reads them as a "remarkably faithful echo of Hegel, who in the 1820s told his students that no one should trust a principle according to which 'things will adjust, they will take care of themselves.' " Also see the epigraphs and Chapter 9 of Carter (2020). As we shall see in Sect. 6 below, these epigraphs serve as the leitmotif for this entire essay.

[2] Notice that it is this convergence in finite periods that distinguishes *eventual* periodicity from *asymptotic* periodicity.

[3] The model of equilibrium growth in its original form dates back to Joan Robinson's 1956 book on *The Accumulation of Capital;* see Deng et al. (2019) for a textual justification of the RSL appellation. For a more recent variant in the equilibrium growth setting, see Tobin (1989).

[4] The RSL model of equilibrium growth has been studied, following Shinkai (1960), by Uzawa (1961, 1963), Takayama (1963), Inada (1963, 1964), and Amano (1964) among others. The textbook treatments of Burmeister and Dobell (1970) and Dixit (1976) remain fresh and up-to-date. The RSL model of optimal growth has been extensively studied in various special cases in Nishimura and Yano (1995, 1996), Khan and Mitra (2006), Fujio (2005, 2006, 2008, 2009) and Fujio and Khan (2006).

[5] See in particular Tobin (1989) and the subsequent discussion by Steedman (1990).

equilibrium trajectories exhibit eventual periodicity but are nevertheless chaotic, we highlight possibilities that are separately well-understood but have never been brought together in the same model.[6] It is satisfying that this can be done in the canonical RSL model, a workhorse in economic dynamics.

Interestingly, for eventual periodicity to arise in equilibrium growth, we demonstrate that the economy almost surely does not diversify its production in the long run. Once the economy enters a period-p cycle, it accumulates capital stock by fully specializing in investment good production for $(p-1)$ periods and then decumulates capital stock by fully specializing in consumption good production for *one* period alone. By carefully picking the depreciation rate, one-period capital decumulation will bring the economy back to the initial capital stock, thus leading to a period-p cycle.

Beyond equilibrium growth, we carry our inquiry further to study the RSL model of optimum growth. In particular, we demonstrate that the building block of eventual periodicity in the form of the Nathanson-Khan-Rajan (henceforth, NKR) construction, a Z-shaped map in the trapping square, is not optimal for any discount factor. Our results shed new light on the sources of chaotic dynamics and, more importantly, their interplay. As identified and synthesized in Mitra et al. (2006), for chaos to arise from a policy function, economic models need to incorporate either upward or downward inertia. Upward inertia refers to the assumption that the state variable of an economy cannot jump upwards instantaneously,[7] while downward inertia refers to the state variable not being able to jump downwards instantaneously.[8] Our construction of eventual periodicity hinges on the marriage of the two types of inertia. The phenomenon of upward inertia meeting downward inertia in the trapping square leads to a unique Z-shaped map obtained through the Keynesian-type savings behavior and is precluded when inter-temporal arbitrage conditions are explicitly brought into play. This result stands in sharp contrast to the aforementioned cases that only one type of inertia is activated—both the check and the tent map can be sustained as optimal under certain discount factor, as in Nishimura and Yano (1995) and Khan and Mitra (2012), but the Z-shaped map, as a combination of the two, cannot be so sustained.

We present the setup of the RSL model of equilibrium growth in the next section. We characterize equilibrium growth in Sect. 3 and discuss the emergence of eventual periodicity in Sect. 4. We then show that the Z-shaped map is not optimal in an optimum growth setting in Sect. 5. We offer concluding remarks in Sect. 6. All the detailed proofs are provided in Sect. 7.1.

[6] See Sorger (1994) and Khan and Piazza (2011) for periodicity; Benhabib and Day (1982), Day and Shaffer (1985), and Nishimura and Yano (1994) for chaotic dynamics.

[7] For example, accumulation of capital stock requires investment and therefore takes time. See the highlighted left arm in Fig. 1 below which plots the capital stock in the next period against the capital stock in the current period. As in Nishimura and Yano (1995), the policy function is a tent map, consisting of an upward-sloping arm that is precisely driven by upward inertia and a downward-sloping arm in the interior of the transition possibility set. A similar construction, albeit in an equilibrium growth framework, can be found in Matsuyama (1999).

[8] For example, investment goods are durable with a depreciation rate strictly below one; see the highlighted right arm in Fig. 1. As in Khan and Mitra (2005), due to this assumption, the right arm of their check-map is upward sloping, which gives rise to topological chaos.

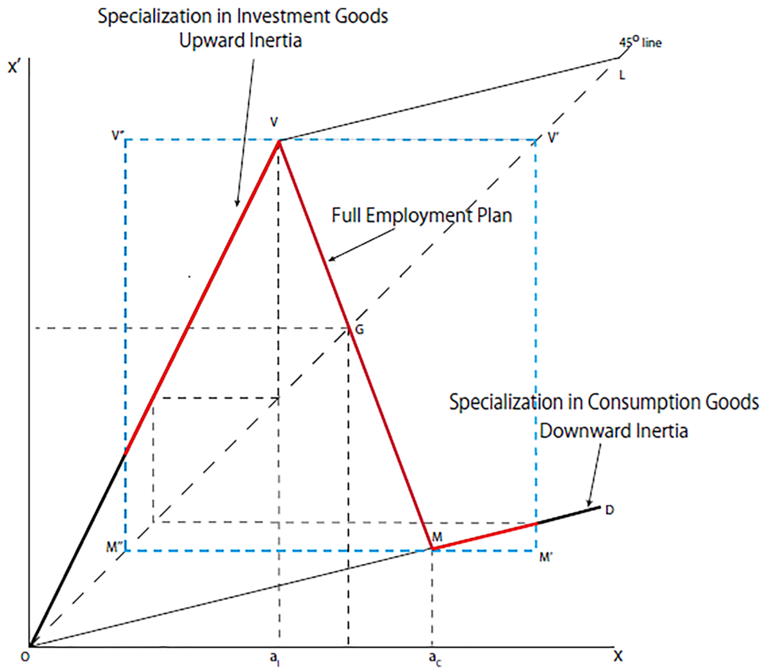

Fig. 1 Eventual periodicity for $p = 3$

2 The two-sector RSL model of equilibrium growth

We consider a two-sector economy. The production technology employs the same Leontief technological specification as in Nishimura and Yano (1995, 1996) and Fujio (2008, 2009). There are two production sectors. One unit of consumption good is produced by one unit of labor and $a_C > 0$ units of capital; $b > 0$ units of investment goods are produced by one unit of labor and $a_I > 0$ units of capital.

Denote by y_C the output of consumption goods and by y_I the output of investment goods. According to the technological specification, we have the following resource constraints for each period:

$$y_C + y_I/b \leq 1 \tag{1}$$

$$a_C y_C + a_I y_I/b \leq x, \tag{2}$$

where the total labor is assumed to be unity and stay constant over time; the total capital stock is given by x.[9]

The product markets for consumption and investment goods are assumed to be perfectly competitive. Under the Leontief production technology, perfect competition implies that

[9] Notice that x, y_C, and y_I may change over time, but for simplicity, we drop the time subscript when presenting the cross-sectional setting of the model.

$$p_C \le w + a_C r \tag{3}$$
$$p_I \le w/b + a_I r/b, \tag{4}$$

where p_C is the price of consumption goods; p_I is the price of investment goods; w is the wage rate; r is the rental rate. We normalize p_C to be unity.[10] The inequality for p_C holds strictly only if $y_C = 0$ and the inequality for p_I holds strictly only if $y_I = 0$. Moreover, $w = 0$ if there is unemployment of labor, i.e., $y_C + y_I/b < 1$; $r = 0$ when there is excess supply of capital, i.e., $a_C y_C + a_I y_I/b < x$. We rewrite the conditions above in a compact form

$$(y_C + y_I/b - 1)w = 0 \tag{5}$$
$$(a_C y_C + a_I y_I/b - x)r = 0 \tag{6}$$
$$(p_C - w - a_C r)y_C = 0 \tag{7}$$
$$(p_I - w/b - a_I r/b)y_I = 0. \tag{8}$$

Denote the savings rate of the capital income by s_I and the savings rate of labor income by s_C.[11] Market clearing conditions yield

$$p_C y_C = (1 - s_C)w + (1 - s_I)rx \tag{9}$$
$$p_I y_I = s_C w + s_I rx. \tag{10}$$

Our setting encompasses several benchmark specifications in the literature of economic growth as special cases. For example, under “extreme classical saving” specification as in Hahn and Matthews (1964), capitalists save all their income by only purchasing investment goods and laborers use all their income for consumption goods, or equivalently, $s_I = 1$ and $s_C = 0$. Under the standard Keynesian setting, a constant fraction of the aggregate income is saved, which coincides with the case $s_C = s_I$.

Conditions (1)–(10) characterize a temporary equilibrium for each period.

The dynamics of the model is generated by capital accumulation. Capital cannot be consumed and depreciates at the rate $d \in (0, 1)$. The amount of capital available at the beginning of next period x' is equal to the sum of the current production of investment goods and the left-over capital after depreciation:

$$x' = (1 - d)x + y_I.$$

We further define a key parameter

$$\zeta \equiv b/(a_C - a_I) - (1 - d) \tag{11}$$

[10] This precludes the possibility of $p_C = 0$, which rules out the case with all factor and product prices being zero.

[11] For similar savings specifications, see chapter 6 of Dixit (1976).

which features prominently in the analysis of the two-sector RSL model of optimum growth as in Deng et al. (2019). ζ can be interpreted as the marginal rate of transformation of capital between the present and the next period under full utilization of resources.

3 The temporary equilibrium and the resulting dynamics

We first define two cutoff capital stocks

$$x_L = \frac{a_I a_C}{s_I a_C + (1 - s_I) a_I} \tag{12}$$

$$x_K = s_C a_I + (1 - s_C) a_C. \tag{13}$$

By construction, we have $\min\{a_C, a_I\} \le x_i \le \max\{a_C, a_I\}$ for $i = L, K$. We now demonstrate that x_L and x_K serve as important benchmarks to delineate the temporary equilibrium of the model.

Lemma 1 *For $x < x_L$, there exists a temporary equilibrium such that there is excess labor: $y_C + y_I/b < 1$. For $x \ge x_L$, if there exists a temporary equilibrium, then labor is fully employed in the equilibrium: $y_C + y_I/b = 1$.*

Lemma 2 *For $x > x_K$, there exists a temporary equilibrium such that there is excess capital: $a_C y_C + a_I y_I/b < x$. For $x \le x_K$, if there exists a temporary equilibrium, then capital is fully utilized in the equilibrium: $a_C y_C + a_I y_I/b = x$.*

The first two lemmas formally establish two very intuitive ideas: the labor constraint of the economy is binding if and only if labor is sufficiently scarce, while the capital constraint of the economy is binding if and only if capital is sufficiently scarce. Our next lemma shows that full utilization of resources is possible in the equilibrium if and only if the capital-labor ratio is at some intermediate range.

Lemma 3 *There exists a temporary equilibrium such that both capital and labor are fully utilized if and only if the capital stock x is in $[\min\{x_L, x_K\}, \max\{x_L, x_K\}]$.*

To ease the exposition of the main equilibrium characterization result and our subsequent analysis of the dynamics, we define

$$\theta \equiv \frac{s_I b}{a_I} + 1 - d.$$

Proposition 1 *If $x_L > x_K$, the equilibrium law of motion of capital is given by*

$$g(x) = \begin{cases} \{\theta x\} & \textit{for } x \in (0, x_K) \\ \left\{\theta x,\ \frac{a_C b}{a_C - a_I} - \zeta x,\ (1-d)x + s_C b\right\} & \textit{for } x \in [x_K, x_L] \\ \{(1-d)x + s_C b\} & \textit{for } x \in (x_L, \infty). \end{cases} \tag{14}$$

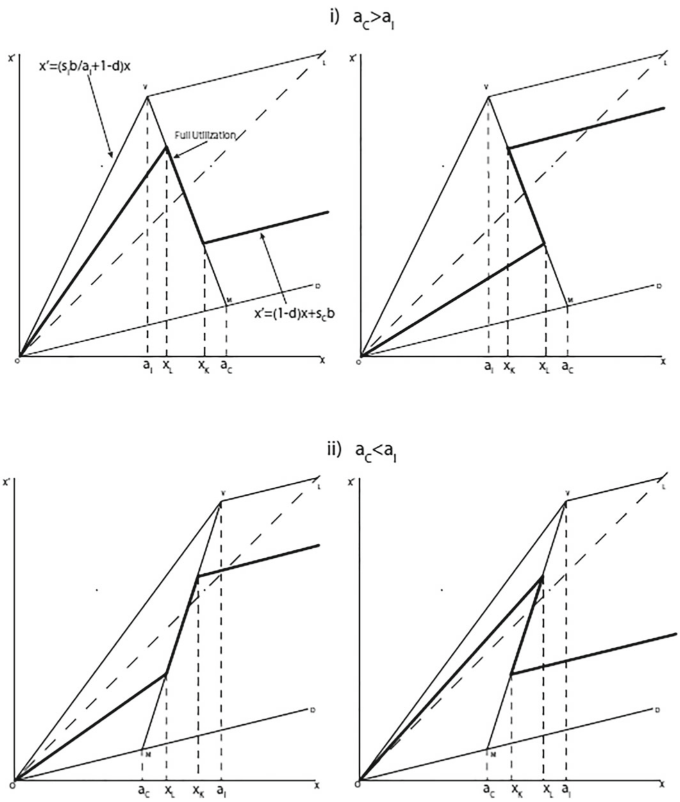

Fig. 2 Characterization of equilibrium dynamics

If $x_L \leq x_K$ and $a_C \neq a_I$, the equilibrium law of motion of capital is given by

$$g(x) = \begin{cases} \theta x & \textit{for } x \in (0, x_L) \\ \frac{a_C b}{a_C - a_I} - \zeta x & \textit{for } x \in [x_L, x_K] \\ (1-d)x + s_C b & \textit{for } x \in (x_K, \infty). \end{cases} \tag{15}$$

If $x_L \leq x_K$ and $a_C = a_I$, the equilibrium law of motion of capital is given by

$$g(x) = \begin{cases} \{\theta x\} & \textit{for } x \in (0, a_I) \\ [(1-d)a_I + \min\{s_I, s_C\}b, & \\ \ (1-d)a_I + \max\{s_I, s_C\}b] & \textit{for } x = a_I \\ \{(1-d)x + s_C b\} & \textit{for } x \in (a_I, \infty). \end{cases} \tag{16}$$

The characterization result can be best seen from Fig. 2 which in each panel the equilibrium growth path is highlighted within the transition possibility set. We illustrate the equilibrium dynamics for two cases: i) $a_C > a_I$ and ii) $a_C < a_I$. We include Fig. 7 in the Appendix to illustrate a special case of $a_C = a_I$ which resembles a one-sector model. Focusing on the top panel of the consumption good sector being capital intensive ($a_C > a_I$) from which our eventual periodicity result will arise, each map consists of generically three arms, capturing the cases of excess labor, full utilization of resources, and excess capital. When $x_L < x_K$, the temporary equilibrium is always unique, and thus the equilibrium dynamics is represented by a function as in Eq. 15. When $x_L > x_K$, due to multiple equilibria, the equilibrium law of motion of capital becomes a correspondence and the three arms occur simultaneously for the interval (x_K, x_L) as suggested by Eq. 14. Furthermore, as in Eqs. 14–16, an increase in s_I shifts the left arm upwards while an increase in s_C shift the right arm upwards.

Being a complete characterization of the equilibrium growth, this result applies to the consumption good sector being either capital intensive ($a_C > a_I$) or labor intensive ($a_C < a_I$). The latter has been considered in Burmeister and Dobell (1970), albeit in a continuous-time setting.[12] When $a_C < a_I$, $\zeta < -1$, the existence of multiple equilibria, which may give rise to non-convergent paths, echoes their earlier discussion. With the same cross-sectional specification, Corden (1966) studies the dynamics of the model in a continuous-time setting without depreciation. Full utilization of labor and capital is the central focus of that paper. In contrast, our results have shown that it is essential to move beyond the case of full utilization for a better understanding of the global dynamics of the model. Moreover, consider the Keynesian savings specification, $s_I = s_C$. According to our definitions, if $s_I = s_C$, then $x_K \geq x_L$, with equality if and only if $a_C = a_I$. Then, the proposition suggests that the Keynesian saving always leads to a unique temporary equilibrium.

Before turning to analyzing the dynamics of the model, we define the following map, $H : \mathbb{R}_+ \to \mathbb{R}_+$, by:

$$H(x) = \begin{cases} (b/a_I + 1 - d)x & \text{for } x \in (0, a_I) \\ a_C b/(a_C - a_I) - \zeta x & \text{for } x \in [a_I, a_C] \\ (1-d)x & \text{for } x \in (a_C, \infty). \end{cases} \tag{17}$$

We refer to H the"Z-map" as its graph is shaped like the alphabet Z. The map has two kinks: one is its top point where x is equal to a_I and the other is its bottom point where x is equal to a_C. The three arms of the Z-map have simple economic interpretations: the left arm stands for full specialization in the sector of investment goods; the right arm stands for full specialization in the sector of consumption goods; the middle arm stands for full utilization of labor and capital. In the analysis of the optimal dynamics of the RSL model as in Fujio (2009), the Z-map stands for the period-by-period zero value loss line, which plays an important role in determining the optimal policy.

Corollary 1 *Consider the extreme classical saving specification: $s_C = 0$ and $s_I = 1$. Let $a_C > a_I$. Then the equilibrium dynamics of the model is represented by the map H.*

[12] See pp. 149–151 in chapter 5 of their book. They impose the extreme classical saving assumption.

4 On eventual periodicity in equilibrium growth

In this section, we provide sufficient conditions, which nest the extreme classical saving specification and the associated Z-map as a special case, for the equilibrium dynamics of the model to exhibit "well-behaved" chaos, a seemingly paradoxical phenomenon first identified by Nathanson (1977) and recently extended by Khan and Rajan (2017).

We first show that under certain parameter restrictions, the equilibrium dynamics of the model is represented by a Z-shaped map. We then follow the NKR construction to show that for any integer $p > 2$, we can always construct a two-sector model of equilibrium growth such that the equilibrium dynamics converges to a p-period cycle from almost everywhere.

Lemma 4 *Let* $I \equiv [(1-d)x_K + s_C b, \theta x_L]$. *For any* $x \in \mathbb{R}_+$, *there is* $t' \in \mathbb{N}$ *such that* $g^t(x) \in I$ *for all* $t > t'$, *if all of the following conditions are satisfied: (i)* $x_L < x_K$*; (ii)* $\theta x_L > x_K$*; (iii)* $(1-d)x_K + s_C b < x_L$. *Moreover, the graph of* g *in* $I \times I$ *is Z-shaped.*

Remark 1 If (i), (ii), and (iii) are satisfied, then

$$\zeta = \frac{g(x_L) - g(x_K)}{x_K - x_L} = \frac{(s_I b/a_I + 1 - d)x_L - (1-d)x_K - s_C b}{x_K - x_L} > 1.$$

Theorem 1 *Let* $p > 2$ *be an integer. Let* $s_C = 0$ *and* $s_I > 0$. *There exists a quadruple* (a_C, a_I, b, d) *such that for almost every real number* $x \in \mathbb{R}_+$, *there corresponds a positive integer* n_x *such that* $g^{n_x}(x)$ *is a point of period* p *for* g.

To see the significance of this result, we consider $p = 3$ with an extreme classical saving specification, $s_I = 1$ and $s_C = 0$. According to this special case of the theorem, we can construct an economy in which the equilibrium dynamics converge from almost everywhere to a three-period cycle: this is plotted in Fig. 1. It is well known that Li–Yorke chaos could be unobservable, but what our result adds further is that the chaos in this constructed economy is not only unobservable but also as well-behaved as one can dream of—it is eventually periodic from almost everywhere!

There are two crucial steps in our construction. First, we pick s_I, a_I, a_C, and b such that all the three conditions in Lemma 4 are satisfied and thus the graph of g is Z-shaped in the trapping square. In particular, to ensure that $\theta x_L > x_K$, the savings rate of capital income s_I needs to be sufficiently high.[13] Second, we pick d such that $(1-d) = \theta^{1-p}$, which implies that $(1-d)$ decreases with p, or equivalently, d increases with p: A higher period p is accompanied with higher depreciation rate.

The intuition behind this result can be better seen from Figs. 3 and 4. Consider an initial stock x_0 in the MM' region, $[x_K, \theta x_L]$. We know $g(x_0) = (1-d)x_0$. Further, given the parameter restrictions in our construction, we know $g(x_0), g^2(x_0), \ldots, g^{p-1}(x_0)$

[13] In our proof, to simplify our construction, we require $s_I > \frac{a_C}{b}$, but an inspection of our argument suggests that it suffices to have $s_I > \frac{a_C - (1-d)a_I}{b}$, or equivalently $\theta a_I > a_C$.

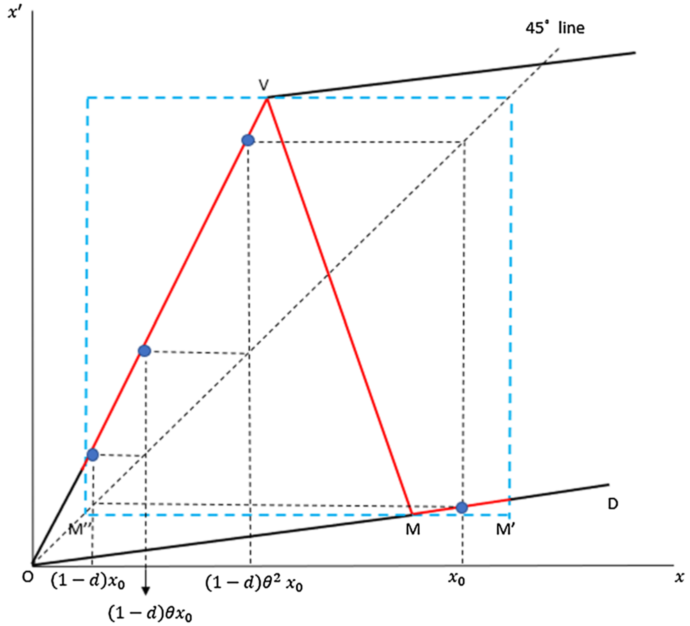

Fig. 3 Eventual periodicity for $p = 4$ ($\theta = 2$, $d = 7/8$)

are all in the $M''V$ region, and given $(1-d) = \theta^{1-p}$, $g^p(x_0) = (1-d)\theta^{p-1}x_0 = x_0$. Since x_0 is arbitrarily chosen from the MM' region, any point in $[x_K, \theta x_L]$ is a periodic point of period-p. By construction, the absolute value of slope of the middle arm VM, ζ, is greater than one, so the fixed point is not stable. Since the economy tends to diverge from the middle arm, as shown by Khan and Rajan (2017), if the economy starts from the middle arm VM, it will almost surely visit the right arm in finite periods, thus leading to the convergence to a period-p cycle. Last, if the economy starts from the left arm, it will always visit either the middle or the right arm, again leading to the convergence to a period-p cycle.

Moreover, evidently in Figs. 3 and 4, once the economy enters a period-p cycle, it alternates between two specialization regimes. The economy accumulates capital stock by fully specializing in producing investment goods for $(p-1)$ periods and then decumulates capital stock by only producing consumption goods for one period. Strong depreciation requires more periods of capital accumulation, resulting in the aforementioned positive relationship between d and p. Almost surely, the resource is not fully utilized in the long run. Borrowing the terminology from Mitra et al. (2006), the two specialization regimes stand for two main sources of chaos: the OV line along which the economy specializes in investment good production corresponds to upward inertia, impossibility of capital stock jumping from zero to a large amount ($a_I > 0$), and the OD line along which the economy specializes in consumption good production corresponds to downward inertia, impossibility of capital stock dropping immediately from a large amount to zero ($d < 1$). The existing work has shown that each of these two types of inertia, combined with the policy function being decreasing for the

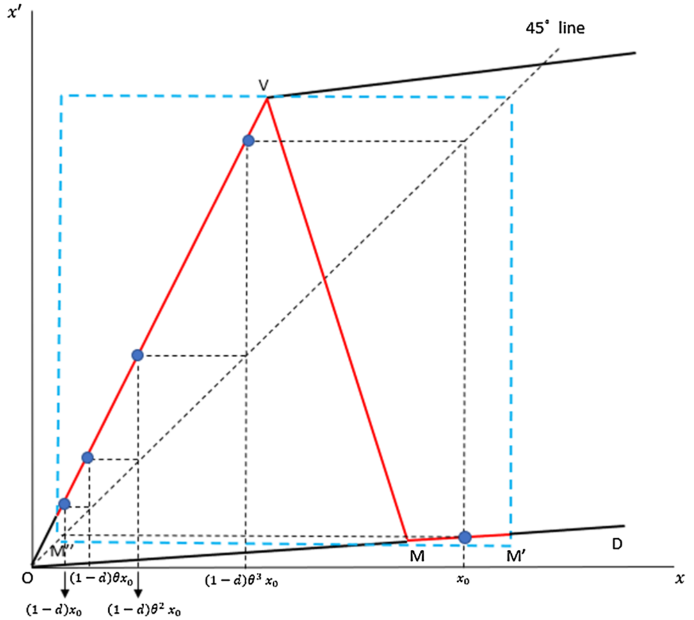

Fig. 4 Eventual periodicity for $p = 5$ ($\theta = 2$, $d = 15/16$)

interior part of the transition possibility set, could give rise to chaos. Our construction shows how chaos could emerge in the form of eventual periodicity when the two types of inertia meet each other in the trapping square.

As a technical note, it should be pointed out that $s_C = 0$ and $s_I > 0$ are also necessary for the NKR construction. In their construction, the left and right arms of the map always intersect on the 45° line from the origin if extended backwards. The assumption $s_C = 0$ is crucial for guaranteeing this property for the map g. To see this, we calculate the intersection point of the left and right arms by solving

$$x' = (s_I b/a_I + 1 - d)x$$
$$x' = (1 - d)x + s_C b.$$

The intersection point is given by $(x, x') = (a_I s_C/s_I, (1-d)a_I s_C/s_I + s_C b)$, which is on the 45° line if and only if (a) $s_C = 0$ or (b) $s_I b/a_I + (1 - d) = 1$. If (b) is satisfied, then the left arm overlaps with the 45° line and hence, Condition (ii) and (iii) in Lemma 4 cannot be jointly satisfied. Therefore, we must have (a) $s_C = 0$ to guarantee that if we extend the left and right arms backwards, they intersect on the 45° line and therefore, the theorem in Khan and Rajan (2017) is applicable. Moreover, $s_I > 0$ is also necessary for the result because otherwise again Condition (ii) and (iii) in Lemma 4 cannot be jointly satisfied. Figure 5 illustrates an example of a Z-shaped map that satisfies $s_C = 0$ and $s_I > 0$, while Fig. 6 plots an example of a Z-shaped map in which the two downward-sloping arms do not intersect on the 45° line.

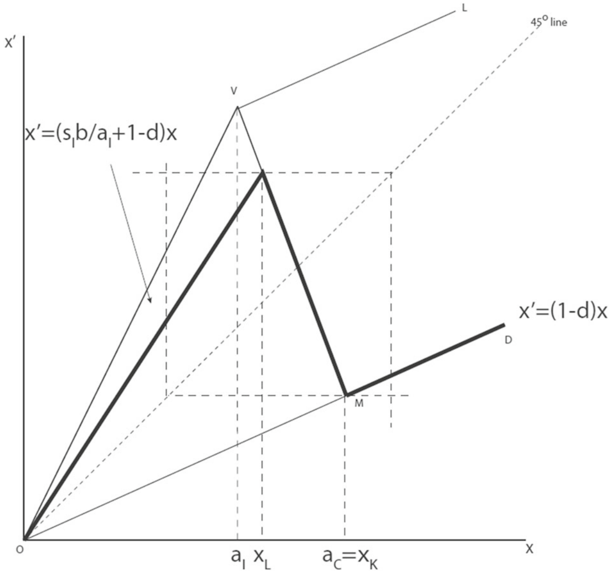

Fig. 5 The Z-shaped map: example I

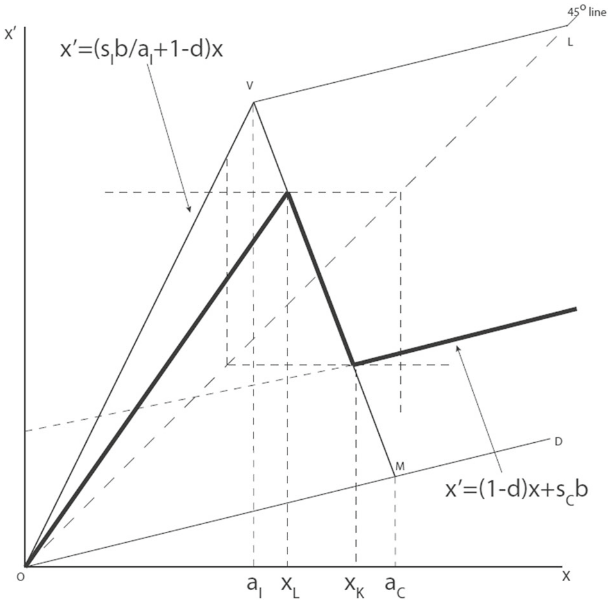

Fig. 6 The Z-shaped map: example II

5 On eventual periodicity in optimum growth

In this section, we demonstrate that eventual periodicity that follows as a characteristic consequence of the NKR construction does not arise in the RSL model of optimum growth. In particular, we show that the Z-map H does not represent the optimal dynamics if the necessary conditions for eventual periodicity as in Lemma 4 are satisfied. It should nevertheless be emphasized that our finding of the incompatibility of one construction of eventual periodicity with intertemporal optimization does not necessarily imply the absence of eventual periodicity in general.

Since the Z-map can only arise from the RSL model of optimum growth with consumption goods being capital intensive ($a_C > a_I$), we consider the same setup as in Deng et al. (2019). The production specification and the law of motion of capital follows the RSL model of equilibrium growth. The transition possibility set is given by

$$\Omega = \{(x, x') \in \mathbb{R}_+ \times \mathbb{R}_+ : x' - (1-d)x \geq 0, x' - (1-d)x \leq b \min\{1, x/a_I\}\},$$

where $\mathbb{R}_+$ is the set of non-negative real numbers. The correspondence of consumption good production is defined as $\Lambda : \Omega \longrightarrow \mathbb{R}_+$ with

$$\begin{aligned}\Lambda(x, x') = \{y \in \mathbb{R}_+ : 0 \leq y \leq (1/a_C)(x - (a_I/b)(x' - (1-d)x))\\ \text{and } \ 0 \leq y \leq 1 - (1/b)(x' - (1-d)x)\}.\end{aligned}$$

The felicity function, $w : \mathbb{R}_+ \longrightarrow \mathbb{R}$, is defined as, $w(y) = y$, and the reduced form utility function, $u : \Omega \longrightarrow \mathbb{R}_+$, is defined as $u(x, x') = \max\{w(y) : y \in \Lambda(x, x')\}$. The discount factor is denoted by $\rho \in (0, 1)$. A program from x_0 is a sequence $\{x_t, y_t\}$ such that for all $t \in \mathbb{N}$, $(x_t, x_{t+1}) \in \Omega$ and $y_t = \max \Lambda(x_t, x_{t+1})$. A program $\{x_t^*, y_t^*\}$ from x_0 is said to be *optimal* if

$$\sum_{t=0}^{\infty} \rho^t [u(x_t, x_{t+1}) - u(x_t^*, x_{t+1}^*)] \leq 0$$

for every program $\{x_t, y_t\}$ from x_0. Denote the optimal policy correspondence by h.

Proposition 2 *If $\rho a_C(1-d) < a_I$, then for any $x \in (0, a_C/(1-d))$ and $x' \in h(x)$, $x' \leq a_C$.*

Corollary 2 *If $a_C(1-d) < a_I$, then the Z-map H is not optimal.*

For the Z-map to be Z-shaped in the trapping square, we must have $a_C(1-d) < a_I$. However, the corollary suggests that precisely because of this necessary condition, the Z-map is no longer optimal. Therefore, intertemporal optimization precludes eventual periodicity in the form of NKR construction. Proposition 2 is based on the characterization of the optimal dynamics in Deng et al. (2019), but its implication is of substantive interest, and of somewhat decisive importance: In a model that marries downward inertia with upward inertia, a map that combines the tent and check-map, the two signature maps characterizing the optimal dynamics in the presence of only upward or downward inertia, is no longer optimal.

6 Concluding remarks

The results reported in this essay underscore a methodological turn in macroeconomics and growth theory: a move from Ramsey to Keynes. As such, by reactivating a canonical model of equilibrium growth, our analysis has interest that goes beyond a narrow technical investigation to the broader issues of comparative and transition dynamics that underlie the notion of *eventual periodicity* investigated here.

> Conventional economic theory rarely if ever faces this crucial question. An analytical solution is obtained, and an equilibrium solved for a certain set of parameters and exogenous variables values. A change is posited, and a new equilibrium solution calculated. The time taken between these equilibria is almost never considered as an issue. A brilliant exception to this was published as early as 1969 by Atkinson.[14]

It is not whether Euler–Lagrange conditions are a prerequisite for the definition of an equilibrium, but rather the difference between a good and bad equilibrium, and the adjustment an equilibrium. This normative property is the issue. And it is here that *eventual periodicity* moves the literature forward by shifting the issue from convergence to convergence in finite time. But to be sure, for the long run to be fully erased from the discussion, one needs an investigation of the (i) robustness of the model, including possibly variable discount rates, (ii) response of the finite time to changing parameters – a deeper quantitative investigation of Atkinson's question as to how long is the long run that the specific parameterization of the RSL model makes possible.[15]

Circling back from Keynes to Ramsey, and moving to specifics, we then show the absence of eventual periodicity in the form of the NKR construction in the RSL model of optimum growth. Could eventual periodicity arise in a different form? To be more specific, Deng et al. (2019) offer a detailed characterization of the optimal dynamics under the parameter restriction of $\zeta \leq 1$,[16] which leads to global convergence to the steady state or two-period cycles. For $\zeta > 1$, the optimal policy of the RSL model has not been fully characterized, but in the light of our result in Sect. 5, it is reasonable to conjecture that optimal dynamics is *not* eventually periodic. . Moreover, in the setting of equilibrium growth, our eventual periodicity result hinges on the condition $\theta^{p-1}(1-d) = 1$. What about the dual condition $(1-d)^{p-1}\theta = 1$? In the

[14] See Ormerod and Rosewell (2006, p. 134) referring to Atkinson (1969). For the recent surveys, see Hammond and Rodriguez-Clare (1993) and Turnovsky (2003); also see Dixit (1970). The authors continue "He showed, inter alia, that the typical time scale of transition from one equilibrium growth path to another in the Solow model was over 100 years. But this article appears to have been exorcised from student reading lists; its implication that economies, even in a strictly neo-classical world, spend a long time out of equilibrium presumably being too disturbing."

[15] We are indebted to the valuable comments of an anonymous and critical referee.

[16] ζ is the the marginal rate of transformation of capital goods today into capital goods tomorrow, defined as $\zeta \equiv b/(a_C - a_I) - (1-d)$.

Appendix, we offer an example that draws direct parallelism with Fig. 1. Can the NKR construction be generalized in this alternative setting? We leave all these questions for future investigation.

Availability of data and materials (data transparency) Not applicable.

Code availability Not applicable.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

7 Appendix

7.1 Detailed proofs

Proof of Lemma 1: To prove the first part, we consider

$$y_C = \frac{(1-s_I)x}{a_C},\ \ y_I = \frac{s_I bx}{a_I},\ \ w = 0,\ \ r = \frac{1}{a_C},\ \ p_I = \frac{a_I}{ba_C}.$$

It is easy to show that Conditions (1)–(10) are satisfied provided that $x < x_L$. In particular, the resource constraint for labor is not binding, that is, $y_C + y_I/b < 1$.

To prove the second part, suppose on the contrary there exists an equilibrium such that $y_C + y_I/b < 1$ for $x \geq x_L$. Since $y_C + y_I/b < 1$, we must have $w = 0$. Since $w + a_C r \geq p_C = 1$, $r > 0$, which implies that capital is fully utilized: $a_C y_C + a_I y_I/b = x$. To proceed, there are three cases: (i) $s_I \in (0, 1)$; (ii) $s_I = 0$; (iii) $s_I = 1$. Consider (i) $s_I \in (0, 1)$. Since $r > 0$, market clearing conditions for consumption and investment goods imply that $p_C y_C > 0$ and $p_I y_I > 0$, which further imply $y_C > 0$ and $y_I > 0$. Since $y_C > 0$, $p_C = w + a_C r = 1$. Further, we know $w = 0$, so $r = 1/a_C$. Since $y_I > 0$, $p_I = w/b + a_I r/b$ with $w = 0$, suggesting $p_I = a_I/(ba_C)$. Since we know w, r, and p_I, the market clearing conditions for both goods imply that $y_C = (1-s_I)x/a_C$ and $y_I = s_I bx/a_I$. Consider case (ii): $s_I = 0$. Since $s_I = 0$, according to the market clearing condition of the investment goods, $p_I y_I = s_C w + s_I rx = 0$. If $p_I = 0$, then $p_I < w/b + a_I r/b$, which implies $y_I = 0$. Therefore, $p_I y_I = 0$ implies $y_I = 0$. Since the resource constraint of capital is binding, $y_C = x/a_C$. For $s_I = 0$, what we have obtained can be rewritten as $y_C = (1-s_I)x/a_C$ and $y_I = s_I bx/a_I$. Consider case (iii): $s_I = 1$. Similarly, since $s_I = 1$, according to the market clearing condition of the consumption goods, $p_C y_C = (1-s_C)w + (1-s_I)rx = 0$. Since $p_C = 1$, we must have $y_C = 0$. Since the resource constraint of capital is binding, $y_I = bx/a_I$. For $s_I = 1$, we rewrite y_C and y_I as $y_C = (1-s_I)x/a_C$ and $y_I = s_I bx/a_I$. In sum, we have shown that for any s_I, $y_C = (1-s_I)x/a_C$ and $y_I = s_I bx/a_I$. We then have

$$y_C + y_I/b = \left(\frac{1-s_I}{a_C} + \frac{s_I}{a_I}\right)x = \frac{x}{x_L} \geq 1,$$

where the last inequality follows from $x \geq x_L$. This contradicts to the supposition that $y_C + y_I/b < 1$. We have now obtained the desired conclusion. □

Proof of Lemma 2: To prove the first part, we consider

$$y_C = 1 - s_C, \ \ y_I = bs_C, \ \ w = 1, \ \ r = 0, \ \ p_I = 1/b.$$

It is easy to show that Conditions (1)–(10) are satisfied provided that $x > x_K$. In particular, the resource constraint for capital is not binding, that is, $a_C y_C + a_I y_I/b < x$.

To prove the second part, suppose on the contrary there exists an equilibrium such that $a_C y_C + a_I y_I/b < x$ for $x \leq x_K$. Since $a_C y_C + a_I y_I/b < x$, we must have $r = 0$. Since $w + a_C r \geq p_C = 1$, $w > 0$, which implies that there is full employment of labor: $y_C + y_I/b = 1$. To proceed, there are three cases: (i) $s_C \in (0, 1)$; (ii) $s_C = 0$; (iii) $s_C = 1$. Consider (i) $s_C \in (0, 1)$. Since $w > 0$, the market clearing conditions for consumption and investment goods imply that $y_C > 0$ and $y_I > 0$. Since $y_C > 0$, $p_C = w + a_C r = 1$. Further, we know $r = 0$, so $w = p_C = 1$. Since $y_I > 0$, $p_I = w/b + a_I r/b$ with $r = 0$, suggesting $p_I = 1/b$. Since we know w, r, and p_I, the market clearing conditions for both goods imply that $y_C = 1 - s_C$ and $y_I = s_C b$. Consider case (ii): $s_C = 0$. Since $s_C = 0$, according to the market clearing condition of the investment goods, $p_I y_I = s_C w + s_I r x = 0$. If $p_I = 0$, then $p_I < w/b + a_I r/b$, which implies $y_I = 0$. Therefore, $p_I y_I = 0$ implies $y_I = 0$. Since the resource constraint of labor is binding, $y_C = 1$. For $s_C = 0$, what we have obtained can be rewritten as $y_C = 1 - s_C$ and $y_I = s_C b$. Consider case (iii): $s_C = 1$. Similarly, since $s_C = 1$, according to the market clearing condition of the consumption goods, $p_C y_C = (1 - s_C) w + (1 - s_I) r x = 0$. Since $p_C = 1$, we must have $y_C = 0$. Since the resource constraint of labor is binding, $y_I = b$. For $s_C = 1$, we can rewrite y_C and y_I as $y_C = 1 - s_C$ and $y_I = s_C b$. In sum, we have shown that for any s_I, $y_C = 1 - s_C$ and $y_I = s_C b$. We then have

$$a_C y_C + a_I y_I/b = a_C(1 - s_C) + a_I s_C = x_K \geq x,$$

which contradicts to the supposition that $a_C y_C + a_I y_I/b < x$. We have now obtained the desired conclusion. □

Proof of Lemma 3: We consider two cases separately: (i) $a_C \neq a_I$; (ii) $a_C = a_I$.

Case (i): $a_C \neq a_I$. We first prove the "only if" part. Suppose on the contrary there exists a temporary equilibrium such that both capital and labor are fully utilized with the capital stock $x \notin [\min\{x_L, x_K\}, \max\{x_L, x_K\}]$. Solving y_C and y_I from the resource constraints, we have

$$y_I = \frac{a_C - x}{a_C - a_I} b, \quad y_C = \frac{x - a_I}{a_C - a_I}. \tag{18}$$

If x is not in $[\min\{a_C, a_I\}, \max\{a_C, a_I\}]$, then either y_I or y_C is strictly negative, which leads to a contradiction. Hence, we must have $x \in [\min\{a_C, a_I\}, \max\{a_C, a_I\}]$. Suppose $x = a_I$. Then $y_C = 0$. Since $y_C = 0$ and we know either $w > 0$ or $r > 0$ from

$p_C \leq w + a_C r$, the market clearing condition of the consumption goods implies that $s_C = 1$ or $s_I = 1$. However, if $s_C = 1$, then $x_K = a_I$; or if $s_I = 1$, $x_L = a_I$. In either case, we have $x = a_I \in [\min\{x_L, x_K\}, \max\{x_L, x_K\}]$, which leads to a contradiction. Therefore, we must have $x \neq a_I$, or according to Eq. (18), equivalently, $y_C > 0$. Since $y_C > 0$, we must have $1 = p_C = w + a_C r$. Together with the market clearing condition of the consumption goods, we obtain two linear equations for w and r

$$w + a_C r = 1 \tag{19}$$

$$(1 - s_C)w + (1 - s_I)xr = \frac{x - a_I}{a_C - a_I}. \tag{20}$$

Solving for w and r, we have

$$w = \frac{(a_I(1 - s_I) + s_I a_C)(x - x_L)}{(a_C - a_I)((1 - s_C)a_C - (1 - s_I)x)} \tag{21}$$

$$r = \frac{x_K - x}{(a_C - a_I)((1 - s_C)a_C - (1 - s_I)x)}. \tag{22}$$

If $x > x_K$ and $x > x_L$, then w and r must take the opposite signs, contradicting to the fact that $w \geq 0$ and $r \geq 0$. Similarly, if $x < x_K$ and $x < x_L$, then w and r must take the opposite signs, again contradicting to the fact that $w \geq 0$ and $r \geq 0$. Therefore, there does not exist a temporary equilibrium in which capital and labor are fully utilized for any $x < \min\{x_L, x_K\}$ or $x > \max\{x_L, x_K\}$.

We now turn to the "if" part. Consider $x \in [\min\{x_L, x_K\}, \max\{x_L, x_K\}]$. Define y_C, y_I, w, and r as in Eqs. (18), (21), and (22). Further, let $p_I = w/b + a_I r/b$. It is easy to show that Conditions (1)–(10) are satisfied. Since by construction, $\min\{a_C, a_I\} \leq x_i \leq \max\{a_C, a_I\}$ for $i = L, K$, $y_I \geq 0$ and $y_C \geq 0$ for $x \in [\min\{x_L, x_K\}, \max\{x_L, x_K\}]$. To see that w and r are non-negative, notice that they are the roots to the Eqs. (19) and (20). For the roots to the two linear equations to be non-negative, it suffices to have

$$\left((1 - s_I)x - \frac{x - a_I}{a_C - a_I}a_C\right)\left((1 - s_C) - \frac{x - a_I}{a_C - a_I}\right) \leq 0,$$

or equivalently,

$$\frac{(x_L - x)(x_K - x)}{(a_C - a_I)^2(s_I a_C + (1 - s_I)a_I)} \leq 0 \Leftrightarrow (x_L - x)(x_K - x) \leq 0,$$

which holds for $x \in [\min\{x_L, x_K\}, \max\{x_L, x_K\}]$. Since $w \geq 0$ and $r \geq 0$, by construction, $p_I \geq 0$. Therefore, we have constructed a temporary equilibrium such that capital and labor are fully utilized.

Case (ii): $a_C = a_I$. In this case, $x_L = x_K = a_C = a_I$. Then $[\min\{x_L, x_K\}, \max\{x_L, x_K\}] = \{a_I\}$. If $x \neq a_I$, it is impossible to find y_I and y_C such that both resource constraints are satisfied. If $x = a_I$, we can pick

$$y_C = 1 - s_C,\ \ y_I = bs_C,\ \ w = 1,\ \ r = 0,\ \ p_I = 1/b,$$

such that all the equilibrium conditions are satisfied. In this equilibrium, both capital and labor are fully utilized. □

Proof of Proposition 1: Consider two cases separately: (i) $x_L > x_K$ and (ii) $x_L \leq x_K$.

Case (i): $x_L > x_K$. Consider $x < x_K$. Since $x < x_K$ and $x_K < x_L$, $x < x_L$. From Lemma 1, there exists a temporary equilibrium with excess labor. According to Lemma 2, the resource constraint of capital in a temporary equilibrium is always binding. According to Lemma 3, there is no equilibrium in which both labor and capital are fully utilized. Therefore, for $x < x_K$, we must have excess supply of labor: $y_C + y_I/b < 1$ and $a_C y_C + a_I y_I/b = x$. Following the proof of Lemma 1, we can show there exists a unique temporary equilibrium with $y_I = s_I bx/a_I$. Using the capital accumulation equation, we have $x' = \theta x$.

Consider $x > x_L$. Since $x > x_L$ and $x_L > x_K$, $x > x_K$. From Lemma 2, there exists a temporary equilibrium with excess capital. According to Lemma 1, the resource constraint of labor is always binding. According to Lemma 3, there is no equilibrium in which both labor and capital are fully utilized. Therefore, for $x > x_L$, we must have excess supply of capital: $y_C + y_I/b = 1$ and $a_C y_C + a_I y_I/b < x$. Following the proof of Lemma 2, there exists a unique temporary equilibrium with $y_I = s_C b$. From the capital accumulation equation, we obtain $x' = (1-d)x + s_C b$.

Consider $x \in (x_K, x_L)$. According to Lemma 1, there exists a temporary equilibrium such that there is excess labor. According to Lemma 2, there exists a temporary equilibrium such that there is excess capital. According to Lemma 3, there exists a temporary equilibrium such that both capital and labor are fully utilized. Following the proofs of Lemmas 1, 2, and 3, y_I can be $s_C b$ (excess capital), $s_I bx/a_I$ (excess labor), or $(a_C - x)b/(a_C - a_I)$ (full utilization of resource). Using the capital accumulation equation, we obtain the desired conclusion.

Consider $x = x_K$. According to Lemmas 1, 2, and 3, there exist only two types of equilibria: excess labor or full utilization of both factors. However, since $(a_C - x)b/(a_C - a_I) = s_C b$ for $x = x_K$, the same expression of x' for $x \in (x_K, x_L)$ applies to the case of $x = x_K$.

Consider $x = x_L$. According to Lemmas 1, 2, and 3, there exist only two types of equilibria: excess capital or full utilization of both factors. However, since $(a_C - x)b/(a_C - a_I) = s_I bx/a_I$ for $x = x_L$, the same expression of x' for $x \in (x_K, x_L)$ applies to the case of $x = x_L$.

Case (ii): $x_L \leq x_K$. Consider $x < x_L$. According to Lemma 1, there exists a temporary equilibrium with excess labor supply. According to Lemma 2, there is always full utilization of capital in the equilibrium. According to Lemma 3, there is no equilibrium with full utilization of resource. Hence, the equilibrium with excess labor supply is unique. Following the proof of Lemma 1, we can show $y_I = s_I bx/a_I$. From the capital accumulation equation, we have $x' = (s_I b/a_I + 1 - d)x$. Consider $x > x_K$. From Lemmas 1, 2, and 3, there is a unique temporary equilibrium with excess capital stock, thus leading to $x' = (1-d)x + s_C b$.

Now consider $x \in [x_L, x_K]$. We consider two subcases: (i) $a_C \neq a_I$; (ii) $a_C = a_I$. For (i), according to Lemmas 1, 2, and 3, there exists a unique temporary equilibrium with full utilization of resource in which y_C and y_I can be uniquely solved, thus leading to $x' = a_C b/(a_C - aI) - \zeta x$. For (ii), according to Lemmas 1, 2, and 3, we know in any temporary equilibrium there is full utilization of resources. Moreover, since $a_C = a_I$, we must have $a_C = a_I = x_L = x_K$, which implies $x = a_I$. It can be seen from the resource constraints that y_C and y_I cannot be uniquely determined in this case. However, we can show that there exists a continuum of equilibria such that $y_I \in [\min\{s_I, s_C\}b, \max\{s_I, s_C\}b]$. To see this, we adopt a similar proof strategy as in the proof of Lemma 3.

Since we know $y_C + y_I/b = 1$, y_C can take value from [0, 1]. Suppose $y_C = 0$, or equivalently, $y_I = b$. The market clearing condition of the consumption goods suggests that $s_I = 1$ or $s_C = 1$. Therefore, we have $y_I = b = \max\{s_I, s_C\}b$. Suppose $y_C > 0$. Collecting the pricing and market clearing conditions of the consumption goods, we obtain

$$w + a_C r = 1$$
$$(1 - s_C)w + (1 - s_I)xr = y_C.$$

We can obtain non-negative roots (w and r) for this two-equation system if and only if

$$(1 - s_C - y_C)((1 - s_I)x - a_C y_C) \leq 0.$$

In other words, a temporary equilibrium exists if and only if the above inequality holds, which can be further simplified as

$$(1 - s_C - y_C)(1 - s_I - y_C) \leq 0,$$

since we know $x = a_I = a_C$. The above inequality holds if and only if $y_C \in [1 - \max\{s_I, s_C\}, 1 - \min\{s_I, s_C\}]$. Following from the resource constraint, this is equivalent to $y_I \in [\min\{s_I, s_C\}b, \max\{s_I, s_C\}b]$. Plugging y_I into the capital accumulation equation, we obtain the desired conclusion. □

Proof of Lemma 4: Since (i) $x_L < x_K$, (ii) $\theta x_L > x_K$, and (iii) $(1-d)x_K + s_C b < x_L$, we have $\theta x_L > x_K > x_L > (1 - d)x_K + s_C b$. Thus, I is nonempty. Next, we pick $x \in \mathbb{R}_+$, and break up the proof into two steps: (1) $x \in I$ and (2) $x \notin I$.

Step (1): $x \in I$. Let's write the interval I as $I = I_1 \cup I_2 \cup I_3$ with $I_1 \equiv [(1-d)x_K + s_C b, x_L]$, $I_2 \equiv [x_L, x_K]$, and $I_3 \equiv [x_K, \theta x_L]$. Consider the first subinterval I_1. From (i) and (ii) we have $\theta x_L > x_K > x_L$ and thus $\theta > 1$, so $g((1 - d)x_K + s_C b) > (1 - d)x_K + s_C b$. Further, since $g(x_L) = \theta x_L$ and g is strictly increasing over I_1, $g(x) \in I$ for any $x \in I_1$. Next, consider the second subinterval I_2. We have $g(x_L) = \theta x_L$ and $g(x_K) = (1 - d)x_K + s_C b$. Further, since g is strictly decreasing over I_2, we have $g(x) \in I$ for any $x \in I_2$. Finally, consider the third subinterval $[x_K, \theta x_L]$. We have

$$\begin{aligned} \theta x_L - g(\theta x_L) &= \theta x_L - (1 - d)\theta x_L - s_C b \\ &> \theta x_L - (1 - d)\theta x_K - s_C b \\ &= \theta(x_L - (1 - d)x_K) - s_C b \\ &> \theta s_C b - s_C b = (\theta - 1)s_C b > 0 \end{aligned}$$

where the first inequality follows from (i), the second from (iii), and the third from $\theta > 1$. We then obtain $g(\theta x_L) < \theta x_L$. Since we know $g(x_K) = (1-d)x_K + s_C b$ and g is strictly increasing over I_3, $g(x) \in I$ for any $x \in I_3$.

Step (2): $x \notin I$. For $x \in (0, (1-d)x_K + s_C b)$, pick t' to be the smallest integer such that $g^{t'}(x) = \theta^{t'} x \geq (1-d)x_K + s_C b$. By construction, $g^{t'-1}(x) < (1-d)x_K + s_C b$, so $g^{t'}(x) < \theta((1-d)x_K + s_C b) < \theta x_L$, which implies that $g^{t'}(x) \in I$. Further, since we have shown $g(x') \in I$ for any $x' \in I$, $g^t(x) \in I$ for any $t > t'$. Similarly, for $x \in (\theta x_L, \infty)$, pick t' to be the smallest integer such that $g^{t'}(x) \leq \theta x_L$. By construction, $g^{t'-1}(x) > \theta x_L$, so $g^{t'}(x) > (1-d)\theta x_L + s_C b > (1-d)x_K + s_C b$, which implies that $g^{t'}(x) \in I$. Since we have shown $g(x') \in I$ for any $x' \in I$, $g^t(x) \in I$ for any $t > t'$.

Notice that $\theta x_L > x_K > x_L > (1-d)x_K + s_C b$, so in $I \times I$, the graph of g has an upward-sloping arm and then a download-sloping middle arm followed by another upward-sloping arm, which resembles the character "Z". □

Proof of Theorem 1: Recall $\theta \equiv s_I b/a_I + (1-d)$. We first fix s_I, a_I, a_C, and b such that $s_I b > a_C > a_I$. Pick d such that

$$1 - d = \theta^{1-p}, \text{ or equivalently, } 1 = (1-d)(s_I b/a_I + (1-d))^{p-1} \equiv \ell(1-d).$$

Since $p > 2$, ℓ is strictly increasing on $(0, 1)$. Further, since $\ell(0) = 0 < 1$ and $\ell(1) = (1 + \frac{bs_I}{a_I})^{p-1} > 1$, there exists a unique $d \in (0, 1)$ such that $1 - d = \theta^{1-p}$ holds. Hence, d is well-defined.

Since $s_C = 0$, $x_K = a_C$. Since $s_I > 0$ and $a_I < a_C$, $a_I \leq x_L < a_C$. Then we have $x_K > x_L$, which is Condition (i) in Lemma 4. Since $s_I b > a_C = x_K$ and $x_L \geq a_I$, $\theta x_L \geq \theta a_I = s_I b + (1-d)a_I > x_K$, which is Condition (ii) in Lemma 4. Since $1 - d = (s_I b/a_I + (1-d))^{1-p}$ and $x_K = a_C$, $(1-d)x_K = \theta^{1-p} a_C < a_C/\theta < x_L$, where the first inequality follows from $p > 2$ and $\theta > 1$, and the second inequality follows from $\theta x_L > x_K = a_C$. Further, since $s_C = 0$, we have $x_L > (1-d)x_K + s_C b$, which is Condition (iii) in Lemma 4. We know from the remark to Lemma 4, Conditions (i), (ii), and (iii) imply $\zeta > 1$. From Lemma 4 and given $s_C = 0$, we restrict our attention to $I = [(1-d)x_K, \theta x_L]$.

To apply the main theorem in Khan and Rajan (2017), we now transform our map to a map defined on $[0, 1]$.[17] Consider a map $f : [0, 1] \to I$, defined as $f(x) = (x_L\theta - x_K(1-d))x + x_K(1-d)$ for x in $[0, 1]$. Let $g_b \equiv f^{-1} \circ g \circ f$. Then we have

$$g_b(x) = \begin{cases} \theta x + k & \text{for } x \in [0, \frac{1-k}{\theta}] \\ -\left(k\frac{\theta^{p-1}-1}{\theta-1} - \frac{1-k}{\theta}\right)^{-1} & \\ \cdot\left(x - k\left(\frac{\theta^{p-1}-1}{\theta-1}\right)\right) & \text{for } x \in [\frac{1-k}{\theta}, k\left(\frac{\theta^{p-1}-1}{\theta-1}\right)] \\ \theta^{1-p}\left(x - k\left(\frac{\theta^{p-1}-1}{\theta-1}\right)\right) & \text{for } x \in [k\left(\frac{\theta^{p-1}-1}{\theta-1}\right), 1]. \end{cases} \tag{23}$$

[17] See the correction of Section 2 of Khan and Rajan (2017) in Remark below.

where $k \equiv \frac{x_K(1-d)(\theta-1)}{x_L\theta - x_K(1-d)}$. By construction, $1-d=\theta^{1-p}$, so

$$\frac{\theta-1}{\theta^p-1} = \frac{\theta-1}{\theta/(1-d)-1} = \frac{(\theta-1)(1-d)a_I}{s_I b} < \frac{x_K(1-d)(\theta-1)}{x_L\theta - x_K(1-d)} = k.$$

To see why the inequality above holds, we note

$$\begin{aligned}\theta x_L = \left(\frac{s_I b}{a_I} + (1-d)\right) x_L &< \left(\frac{s_I b}{a_I} + (1-d)\right) x_K \\ \theta x_L - (1-d)x_K &< \left(\frac{s_I b}{a_I}\right) x_K \\ \frac{a_I}{s_I b} &< \frac{x_K}{\theta x_L - (1-d)x_K},\end{aligned}$$

where the first inequality follows from $x_L < x_K$. Moreover, we know $\theta x_L > x_K$, so $\theta x_L - (1-d)x_K > d x_K$, or equivalently, $\frac{1}{d} > \frac{x_K}{\theta x_L - (1-d)x_K}$, which implies

$$\frac{\theta-1}{\theta^{p-1}-1} = \frac{(\theta-1)(1-d)}{d} > \frac{x_K(1-d)(\theta-1)}{x_L\theta - x_K(1-d)} = k,$$

where the first equality follows from $\theta^{1-p} = 1-d$. Since $p > 2$ and $\theta > 1$, we must have $(\theta-1)/(\theta^p-1) > 0$ and $(\theta-1)/(\theta^{p-1}-1) < 1$. Thus, we have shown that

$$0 < \frac{\theta-1}{\theta^p-1} < k < \frac{\theta-1}{\theta^{p-1}-1} < 1.$$

According to Theorem 2.2 in Khan and Rajan (2017), for almost every real number $x \in [0, 1]$, there corresponds an integer $n_x > 0$ such that $g_b^{n_x}(x)$ is a point of period p for g_b. Further, since f is one-to-one, onto, and continuous, and f^{-1} is continuous, f is homeomorphism. According to Ruette (2018), g_b and g are topologically conjugate. The result above carries over to g and we obtain the desired conclusion. □

Remark 2 Notice that the first inequality concerning the parameter b in Section 2 of Khan and Rajan (2017) should be corrected as

$$0 < \frac{m-1}{m^p-1} < b < \frac{m-1}{m^{p-1}-1} < 1.$$

Moreover, as pointed out by one of our referees, for $p=3$, $f^3(\xi)=\xi$ and hence the inequality in the proof of Lemma 2.1 in their paper should be written as $0 < f^3(\xi) \leq \xi < f(\xi) < f^2(\xi)$.

Proof of Proposition 2: We borrow the follow characterization results for the optimal dynamics from Deng et al. (2019).

Lemma 5 (Deng et al. 2019, Theorem 1) *Let* $\zeta > 1$. *The optimal policy correspondence h satisfies*

$$h(x) \subset \begin{cases} \{\beta x\} & \text{for } x \in (0, \hat{x}/\beta] \\ [\hat{x}, \beta x] & \text{for } x \in (\hat{x}/\beta, a_I] \\ [\hat{x}, \zeta(\hat{x}-x)+\hat{x}] & \text{for } x \in (a_I, \hat{x}] \\ [\zeta(\hat{x}-x)+\hat{x}, \hat{x}] & \text{for } x \in (\hat{x}, a_C] \\ [(1-d)x, \hat{x}] & \text{for } x \in (a_C, \hat{x}/(1-d)) \\ \{(1-d)x\} & \text{for } x \in [\hat{x}/(1-d), \infty) \end{cases},$$

where $\beta \equiv b/a_I + (1-d)$ *and the golden rule stock* $\hat{x} = a_C(\zeta + 1 - d)/(\zeta + 1)$.

If $a_I\beta \leq a_C$, then the result directly follows from Lemma 5. Consider $a_I\beta > a_C$, or equivalently, $a_I > a_C/\beta$. From Lemma 5, we only need to consider x in $(a_C/\beta, \hat{x} - (a_C - \hat{x})/\zeta)$. Suppose on the contrary there exists $x_0 \in (a_C/\beta, \hat{x} - (a_C - \hat{x})/\zeta)$ and $x_1^* \in h(x)$ such that $x_1^* > a_C$. Consider an optimal program, $\{x_t^*\}_{t=0}^{\infty}$ starting from x_0. Since $x_1^* > a_C$, $x_2^* > (1-d)a_C$ due to the feasibility of technology. We know from Lemma 5 that $x_2^* \leq \max\{\hat{x}, (1-d)x_1^*\}$. Then, we can always find a program, starting from the same initial capital stock x_0 takes $x_1' = x_1^* - \varepsilon$ for a sufficiently $\varepsilon > 0$ and the same plan afterwards, i.e., $x_t' = x_t^*$ for $t \geq 2$. For ε small enough, this program is technologically feasible. In order to check whether the latter program overtakes the former it suffices to compare the sum of utilities for the first two periods since these two programs take the same plan after the second period. Then we have

$$\begin{aligned} & u(x_0, x_1^*) + \rho u(x_1^*, x_2^*) - u(x_0, x_1^* - \varepsilon) - \rho u(x_1^* - \varepsilon, x_2^*) \\ &= -\frac{a_I}{a_C b}(x_1^* - x_1^* + \varepsilon) + \rho\frac{1-d}{b}(x_1^* - x_1^* + \varepsilon) \\ &= \frac{1-d}{b}\varepsilon\left(\rho - \frac{a_I}{a_C(1-d)}\right) < 0 \end{aligned}$$

where the inequality follows from $\rho a_C(1-d) < a_I$. This leads to a contradiction and establishes the desired result. □

7.2 Illustration of the equilibrium dynamics when $a_C = a_I$

Figure 7 illustrates equilibrium dynamics under a special case of $a_C = a_I$ which resembles a one-sector model.

7.3 The equilibrium dynamics for the RSS model

When $a_I = 0$, our technological specification follows the Robinson–Solow–Srinivasan (RSS) model. In this case, $x_L = 0$ and $x_K = (1 - s_C)a_C$. With a slight modification of the argument, Proposition 1 applies and the equilibrium dynamics is given by

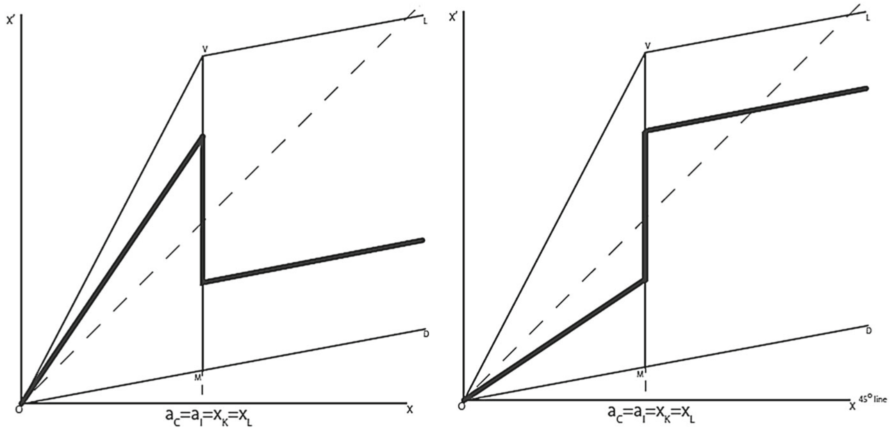

Fig. 7 Characterization of equilibrium dynamics: $a_C = a_I$

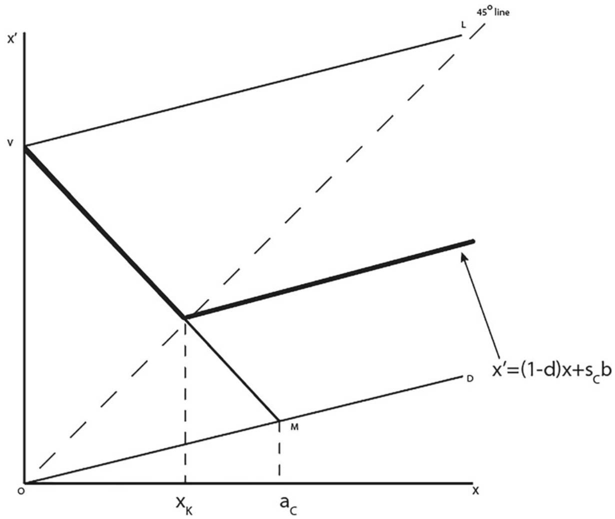

Fig. 8 Equilibrium dynamics: the RSS specification

$$g(x) = \begin{cases} b - \zeta x & \text{for } x \in [0, (1 - s_C)a_C] \\ (1 - d)x + s_C b & \text{for } x \in ((1 - s_C)a_C, \infty). \end{cases} \tag{24}$$

If the dynamics does not lead to convergence, the check-map, as in an optimum growth RSS model with certain restrictions on the discount factor (Khan and Mitra 2012), emerges. Note that under the RSS specification, s_I does not enter the equation for equilibrium dynamics. This is because there are only two types of temporary equilibrium: saving rate of the capital income no longer matters for growth if there is full utilization of resources or if the rental rate is zero under the excess supply of capital. Figure 8 illustrates the equilibrium dynamics of this special case.

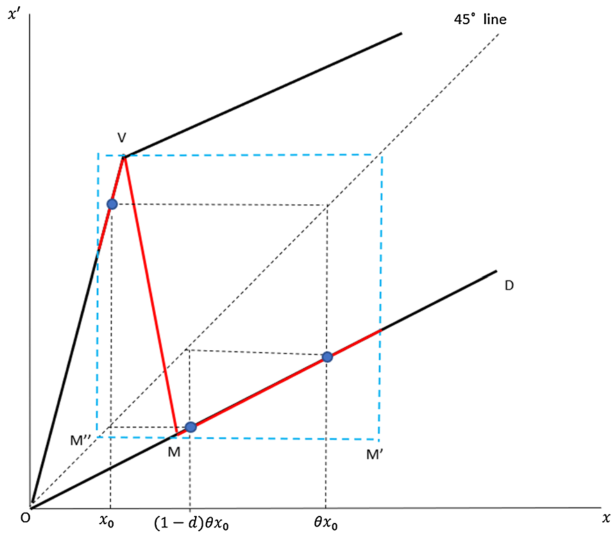

Fig. 9 Equilibrium dynamics for $(1-d)^{p-1}\theta = 1$ ($\theta = 4$, $d = 1/2$)

7.4 Illustration of the equilibrium dynamics when $(1-d)^{p-1}\theta = 1$

Figure 9 illustrates equilibrium dynamics for $(1-d)^{p-1}\theta = 1$ with $p = 3$, $\theta = 4$, and $d = 1/2$. Under the condition of $(1-d)^{p-1}\theta = 1$, any initial stock in the M”V region leads to a period-p cycle.

References

Amano, A.: A further note on Professor Uzawa's two-sector model of economic growth. Rev. Econ. Stud. **31**, 97–102 (1964)

Atkinson, A.: The timescale of economic models: how long is the long run? Rev. Econ. Stud. **36**(2), 137–152 (1969)

Benhabib, J., Day, R.: A characterization of erratic dynamics in the overlapping generation model. J. Econ. Dyn. Control **4**, 37–55 (1982)

Burmeister, E., Dobell, A.R.: Mathematical Theories of Economic Growth. MacMillan, New York (1970)

Carter, Z.D.: The Price of Peace: Money, Democracy and the Life of John Mynard Keynes. Random House, London (2020)

Corden, W.M.: The two sector growth model with fixed coefficients. Rev. Econ. Stud. **33**, 253–262 (1966)

Day, R., Shaffer, W.: Keynesian chaos. J. Macroecon. **7**, 277–295 (1985)

Deng, L., Fujio, M., Khan, M.A.: Optimal growth in the Robinson–Shinkai–Leontief model: the case of capital-intensive consumption goods. Stud. Nonlinear Dyn. Econom. **23**(4), 20190032 (2019)

Dixit, A.: Growth patterns in a dual economy. Oxf. Econ. Pap. **22**(2), 229–234 (1970)

Dixit, A.K.: The Theory of Equilibrium Growth. Oxford University Press, Oxford (1976)

Fujio, M.: The Leontief two-sector model and undiscounted optimal growth with irreversible investment: the case of labor-intensive consumption goods. J. Econ. **86**(2), 145–159 (2005)

Fujio, M.: Optimal transition dynamics in the Leontief two-sector growth model. PhD thesis, The Johns Hopkins University (2006)

Fujio, M.: Undiscounted optimal growth in a Leontief two-sector model with circulating capital: the case of a capital-intensive consumption good. J. Econ. Behav. Organ. **66**(2), 420–436 (2008)

Fujio, M.: Optimal transition dynamics in the Leontief two-sector growth model with durable capital: the case of capital-intensive consumption goods. Jpn. Econ. Rev. **60**(4), 490–511 (2009)
Fujio, M., Khan, M.A.: Ronald W. Jones and two-sector growth: Ramsey optimality in the RSS and Leontief cases. Asia-Pac. J. Account. Econ. **13**(2), 87–110 (2006)
Hahn, F.H., Matthews, R.C.O.: The theory of economic growth: a survey. Econ. J. **74**(296), 779–902 (1964)
Hammond, P.J., Rodriguez-Clare, A.: On endogenizing long-run growth. Scand. J. Econ. **95**(4), 391–425 (1993)
Inada, K.I.: On a two-sector model of economic growth: comments and a generalization. Rev. Econ. Stud. **30**, 119–127 (1963)
Inada, K.I.: On the stability of growth equilibrium in two-sector models. Rev. Econ. Stud. **31**, 127–142 (1964)
Keynes, J.M.: A Tract on Monetary Reform. Macmillan, London (1923)
Khan, M.A., Mitra, T.: On choice of technique in the Robinson–Solow–Srinivasan model. Int. J. Econ. Theory **1**(2), 83–110 (2005)
Khan, M.A., Mitra, T.: Undiscounted optimal growth in the two-sector Robinson–Solow–Srinivasan model: a synthesis of the value-loss approach and dynamic programming. Econ. Theor. **29**(2), 341–362 (2006)
Khan, M.A., Mitra, T.: Impatience and dynamic optimal behavior: a bifurcation analysis of the Robinson–Solow–Srinivasan model. Nonlinear Anal. Theory Methods Appl. **75**(3), 1400–1418 (2012)
Khan, M.A., Piazza, A.: Optimal cyclicity and chaos in the 2-sector RSS model: an anything-goes construction. J. Econ. Behav. Organ. **80**(3), 397–417 (2011)
Khan, M.A., Rajan, A.V.: On the eventual periodicity of piecewise linear chaotic maps. Bulletin of the Australian Mathematical Society **95**(3), 467–475 (2017)
Li, T.Y., Yorke, J.A.: Period three implies chaos. Am. Math. Mon. **82**(10), 985–992 (1975)
Mann, G.: In the Long Run We Are All Dead: Keynesianism, Political Economy and Revolution. Verso, London (2017)
Matsuyama, K.: Growing through cycles. Econometrica **67**(2), 335–347 (1999)
Mitra, T., Nishimura, K., Sorger, G.: Optimal cycles and chaos. In: Dana, R.A., Le Van, C., Mitra, T., Nishimura, K. (eds.) Handbook of Optimal Growth, Chap. 6, vol. 1, pp. 141–169. Springer, Berlin (2006)
Nathanson, M.B.: Permutations, periodicity, and chaos. J. Combin. Theory Ser. A **22**(1), 61–68 (1977)
Nishimura, K., Yano, M.: Optimal chaos, nonlinearity and feasibility conditions. Econ. Theor. **4**(5), 689–704 (1994)
Nishimura, K., Yano, M.: Nonlinear dynamics and chaos in optimal growth: an example. Econometrica **63**(4), 981–1001 (1995)
Nishimura, K., Yano, M.: Chaotic solutions in dynamic linear programming. Chaos Solitons Fractals **7**(11), 1941–1953 (1996)
Ormerod, P., Rosewell, B.: Validation and verification of agent-based models in the social sciences. In: Squazzoni, F. (ed.) Epistemological Aspects of Computer Simulation, pp. 130–140. Springer, Berlin (2006)
Ruette, S.: Chaos on the Interval. Volume 67 of University Lecture Series, American Mathematical Society (2017)
Shinkai, Y.: On equilibrium growth of capital and labor. Int. Econ. Rev. **1**(2), 107 (1960)
Sorger, G.: On the structure of Ramsey equilibrium: cycles, indeterminacy, and sunspots. Econ. Theor. **4**(5), 745–764 (1994)
Steedman, I.: Growth and distribution: a muddling exercise. Camb. J. Econ. **14**(2), 229–232 (1990)
Takayama, A.: On a two-sector model of economic growth: a comparative statics analysis. Rev. Econ. Stud. **30**, 95–104 (1963)
Tobin, J.: Growth and distribution: a neoclassical Kaldor–Robinson exercise. Camb. J. Econ. **13**(1), 37–45 (1989)
Turnovsky, S.: Old and new growth theories: a unifying structure? In: Salvadori, N. (ed.) Old and New Growth Theories: an Assessment, pp. 1–44. Edward Elgar, Cheltenham (2003)
Uzawa, H.: On a two-sector model of economic growth. Rev. Econ. Stud. **29**, 40–47 (1961)
Uzawa, H.: On a two-sector model of economic growth II. Rev. Econ. Stud. **30**, 105–118 (1963)