# On Growing through Cycles: <br> Matsuyama's M-map and Topological Chaos 

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#### Abstract

In this essay, we use the M-map, the functional representation of Matsuyama's (1999) endogenous growth model, as a lever for methodological reflection on differing criteria for chaos and the variety of benchmarks of a complicated dynamical system. This is done through (i) a complete characterization of all iterates of the Mmap for a specific parameter value, (ii) the non-existence of 3- or 5 -period cycles for any admissable parameter value, (iii) the existence of cycles of all other periods for a range of parameter values, and (iv) a bound for the topological entropy of the M-map stemming from the existence of a 7-period cycle. These findings allow comment on the recent numerical-experimental literature, and rely on the first four iterations of the M-map. The formulae for these iterates have independent interest. (130 words) Journal of Economic Literature Classification Numbers: D90, C62, O21. Key Words: Matsuyama's model, Solow regime, Romer regime, cycles, 3-period, 5period, 7-period, iterations, expansive maps, parametric restrictions, KP-construction, Li-Yorke chaos, Devaney chaos, ergodic chaos, topological entropy.


Running Title: M-map revisited

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Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas? ${ }^{1}$

Lorenz (1993)
One establishes the failure of Newtonian determinism by using Newton's own equations. The coin-flipping syndrome is pervasive. "Sensitive dependence on initial conditions" has become a catchword of modern science ${ }^{2}$ Smale (1998)

Triggered by mathematical discovery, the Chaos Revolution is a bifurcation event in the history of the sciences, comprised of sequential paradigm shifts. Perhaps also a major transformation in world cultural history ${ }^{3}$

Abraham (2000)
The term "chaos" was introduced into mathematics by Li and Yorke in 1975 without formally defining what chaos is. Afterwards, various definitions were proposed. They do not coincide in general and none of them can be considered as the unique "good" definition of chaos. One may ask "What is chaos then?" ${ }^{4}$ Ruette (2015)

## 1 Introduction

In a conceptually-imaginative contribution, Matsuyama [39] provided a model that generates endogenous growth through the introduction of new varieties chosen from a continuum of commodities, a technological specification pioneered by Rivera-Batiz-Romer [49]; also see the subsequent discussion in [16], [14], [60] and [62]. A particularly attractive feature of the model is its demonstration, under specific parametric restrictions, that an economy may find itself in a stagnant Solow regime, and a dynamic Romer regime, or at alternative, exogenously-specified equal intervals of time, in both $5^{5}$ From a technical point of view, the model can be represented by the, here so-called, M-map: a piecewise smooth two-parameter map, one of whose arms is the (monotonically-increasing) intensive form of the Cobb-Douglas function, and the other, a (monotonically-decreasing) version of the solution to a differential equation involving the logistic map ${ }^{6]}$ The arms

[^1]are stitched together at a non-differentiable kink, subsequently referred to as the critical point of the map. The model is thus of interest for both substantive and technical reasons.

However, the primary objective of this essay is neither substantive nor technical but methodological. It is to use results on the M-map to interrogate the following prevailing conceptions in economic dynamics:
(a) that we have available a rigorous and mathematically-precise definition of what the noun chaos, and its qualifying adjectives, topological, ergodic and statistical, mean, and that thereby any chaotic dynamical system can be uniquely categorized and unambiguously represented;
(b) that the Li-Yorke theorem pertaining to a "scrambled" set of uncountable cardinality exhausts the meaning of topological chaos, $7^{7}$
(c) that results on the existence of cycles are rendered irrelevant if those cycles are unstable or if the map is "non-generic";
(d) that the "scrambled" set of a map exhibiting Li-Yorke chaos may be (typically is) of measure zero, and thereby unobservable, and that in the light of this "unobservability," there is nothing more to say about topological chaos;
(e) that the success of experimental mathematics and of high-speed computing renders "numerical proofs" substitutable for (formal and traditional) mathematical proofs;
(f) that mathematical results pertaining to smooth non-linear maps carry over, more or less, to smooth maps with kinks or even to linear maps with kinks;
(g) that the notion of a model as an explanation of economic phenomena is epistemologically viable in spite of the (uncountably) many maps with totally diverse and mutually-exclusive dynamical properties that are embedded in a particular model.

We use technical investigations of the M-map presented below to contest and question each of these prevailing conceptions that are now conventional in economic science ${ }^{8}$

However, a secondary objective of the essay are these technical results themselves. Correspondingly, we structure the paper in a loop-like cyclical way. We first present these results in a context totally jettisoned of our methodological preoccupations, and with
$b \exp (-x)$ ), both $a$ and $b$ positive, but a change-of-variable in the M-map leads to at least one of the two parameters being negative; see for example 10 and their reference to the text of Johnson-Kotz, now revised as Johnson-Kotz-Balakrishnan. Also see Robinson's text [50, p. 2].
${ }^{7}$ We explain the technical terms referred to in this, and the subsequent, paragraph in Sections 6 and 7 below.
${ }^{8}$ It is not that one can reductively assert that these conventional understandings are correct or incorrect, but, as we shall argue in Sections 6 and 7 below, that for the future progress of the subject, they need to be qualified and nuanced.
the pioneering analysis of Mitra [42] as the relevant backdrop, single out the M-map for a deeper analytical treatment. Mitra took Matsuyama's observation that the Mmap cannot have a 3-period cycle as a point of departure to develop a weaker sufficient condition for Li-Yorke chaos, one hinging on the relation of the third iterate of the critical point to its unstable fixed point. $\sqrt[9]{ }$ He applied his sufficient condition to establish (also furnish numerical values for) the possibility of the M-map being a Li-Yorke chaotic map. Subsequent work of Mukherji [45], and of Gardini et al. [19], followed in Mitra's footsteps not so much as on Li-Yorke chaos, much less on his sufficient condition for it, but on the stability of 2 - and 4 -period cycles. They also presented a bifurcation analysis based on a comprehensive numerical investigation. Yano-Sato-Furukawa [61] carried the conversation forward by changing the subject to ergodic chaos.

Generalizations of the Li-Yorke criteria parallel to that of Mitra, have been available in the mathematical literature for quite some time, and the questions this work asks and answers have not been posed for the M-map: in the light of the generalization of the Li-Yorke theorem in [46], what about the existence or non-existence of 5- and 7-period cycles? And in the light of [15], what about any odd-numbered cycle? And finally, in the light of [35], what about a cycle of order $2^{n}$ for any natural number $n ? ?^{10}$ More to the point, a rigorous mathematical proof of the non-existence of a 3-period cycle, one that would underpin the motivation for Mitra's influential sufficient condition, has still not been offered for a model that is well on its way to becoming an important and canonical marker for the growth and development as well as the macroeconomic literature ${ }^{11}$

Sections 3 to 5 of this paper addresses this deficiency. They show that 3 - and 5 -period cycles do not exist for any feasible parameter values of the M-map, but cycles of all other periods do exist for specific identifiable intervals of parameter values. In particular, they show that one can identify an interval of parameter values for which a 7 period cycle can exist. All this is executed through a presentation of analytical formulae for the first four iterates of the M-map. These iterate specifications serve to delineate the manifold of the 2-dimensional parameter space, and to complement, through additional geometric and algebraic considerations, the numerical analysis in [19]. In particular, the second iterate is used in Section 2 to give a complete characterization of all iterates for a specific parameter value that allows a 2-period oscillation between the Solow and Romer regimes $\left[^{12}\right.$ We observe in passing that it is a little surprising that such iterates have not been reported in the literature on economic dynamics: they constitute the tertiary (technical) objective of the paper ${ }^{13}$

[^2]With these technical results in hand, we circle back in Sections 6 and 7 to the methodological desiderata listed above and connect to the epigraphs that narrativize the evolution of the chaos story. We try to get some handle on what a chaotic system can possibly mean, and through it, to get some sense of the interesting ideas that are current in the mathematical literature on dynamical systems. Section 8 concludes the paper with some open questions. These sections recount that, even with the limitation to topological chaos, as opposed to statistical and ergodic chaos, one lands into a veritable chaos of different definitions. In its entirety, this paper can then be read as a plea that the word "chaos", both as a noun and as an adjective, used with some circumspection and not be over-indulged: it is after all an understanding of the complexity of a dynamical system that is being sought.

## 2 The M-map and its Second Iterate

This tripartite section specifies the M-map arising from Matsuyama's cyclical growth model, and with the devotion of the second iterate of the M-map and to special case, begins the analytical argumentation that constitutes the paper.

### 2.1 The M-map

The basic model is lucidly diagrammed in [39] (also see Figure 1 presented here), and the algebra of its dynamics is given by

$$
x_{t}=f\left(x_{t-1}\right) \equiv \begin{cases}f_{l}\left(x_{t-1}\right)=G x_{t-1}^{1-1 / \sigma} & 0 \leq x_{t-1} \leq 1  \tag{1}\\ f_{r}\left(x_{t-1}\right)=\frac{G x_{t-1}}{1+\theta\left(x_{t-1}-1\right)} & x_{t-1}>1\end{cases}
$$

where $\theta \equiv(1-1 / \sigma)^{1-\sigma}, \alpha=1-(1 / \sigma)$, leading to $\sigma>1$ and $\alpha \in(0,1)$. We refer the reader to the economic interpretation of these parameters already available in the literature ${ }^{[14}$ and move on to a transformation of variables whereby the M-map can be rewritten in terms of the pair $(G, \beta)$,

$$
f(x)= \begin{cases}f_{l}(x)=G x^{\alpha} & 0 \leq x \leq 1  \tag{2}\\ f_{r}(x)=\frac{G \beta x}{\beta-1+x} & x>1\end{cases}
$$

where $\beta=1 / \theta=\alpha^{\frac{\alpha}{1-\alpha}}$ with $\beta$ in $(1 / e, 1)$ and decreasing with $\alpha$, and $1<G<(1 / \beta)-1$. As brought out in [39, 19], a cyclical growth pattern occurs when $G \in(1,(1 / \beta)-1)$ and
[61], and subsequent work in [52, 53, 54, 62], work with the second iterate, and [45] and [19] diagram numerical snapshots of both the $2^{n d}$ and $4^{t h}$ iterates. Our point simply is that explicit formulae for these iterates of the M-map, as are offered below, have not been furnished and used for analytical results.
${ }^{14}$ In addition to [39] see, for example, [40, 45, 16] and [62, Appendix].
$\beta<1 / 2$ (equivalently, $\alpha>1 / 2$ ). This implies that

$$
\begin{equation*}
f(G)=f^{2}(1)=\frac{\beta G^{2}}{(\beta-1+G)} \equiv \tau<1 \tag{3}
\end{equation*}
$$

With $\tau$ as an important benchmark, we turn to the derivation of the second iterate.

### 2.2 The Second Iterate of the M-map

Consider the M-map given in Figure 1 ( also in Figure 2) and note how the second arm dips below unity to take the value $\tau$. Two other benchmarks are the elements $x_{11}$ and $x_{12}$ of the set $f^{-1}(1)$, and they are obtained by taking at the vertical determined by the critical point $(1, G)$, and through the use of the 45 -degree line, the resulting horizontal to intersect with at the unimodal M-map the two points. Thus, we obtain the four intervals that go towards the determination of the second iterate. We can now algebraically compute it to be

$$
f^{2}(x)= \begin{cases}G^{1+\alpha}(x)^{\alpha^{2}} & \text { if } 0 \leq x \leq x_{11} \equiv G^{-1 / \alpha}  \tag{4}\\ \frac{\beta G^{2} x^{\alpha}}{\beta-1+G x^{\alpha}} & \text { if } x_{11} \leq x \leq 1 \\ \frac{\beta^{2} G^{2} x}{(\beta-1)^{2}+(\beta-1+\beta G) x} & \text { if } 1 \leq x \leq x_{12} \equiv \frac{1-\beta}{1-\beta G} \\ G\left(\frac{\beta G x}{\beta-1+x}\right)^{\alpha} & \text { if } x_{12} \leq x \leq G\end{cases}
$$

Note that $f\left(x_{11}\right)=f\left(x_{12}\right)=1$, leading to $f^{2}\left(x_{11}\right)=f^{2}\left(x_{12}\right)=G$, and that $f^{2}(G)=G \tau^{\alpha}$.
When diagrammed as Figure 1, we note three intersections of the second iterate with the 45 -degree line: the fixed point $\hat{x}$ of the M-map itself, and a 2-period cycle. All this substantiates earlier findings: in particular, we reproduce Theorem 6.1 in Yano-Sato-Furukawa [61] that offers necessary and sufficient conditions for a range of parameter values under which the M-map has the absolute value of its slope greater than unity.
Proposition 1. The M-map, $f:[\tau, G] \rightarrow[\tau, G]$, is expansive, if and only if

$$
\frac{1}{\alpha}<G<\frac{1-\beta}{2}\left[(\beta+2)+\sqrt{\beta^{2}+4 \beta}\right]
$$

We refer the reader to [61, 62] for the proofs and additional discussion and return to expansivity and iterative expansivity of a map in Section 7. We now turn to the case $\tau=1$.

We also invite the reader to compare Figure 1 with Figure 3 in [19], but all the while keeping in mind that we give a global algebraic "picture" as opposed to the local, numerical pictures available there. We emphasize the care that is needed to decipher the local numerical pictures: in Figures 3a and 3b in [19], all potential appearances to the contrary, there is a unique two-period cycle as opposed to a continuum.

### 2.3 A Continuum of 2-period Cycles

Mukherji [45] has focused on 2-period cycles, and investigated the special case when $\beta=1 /(1+G)$ as of particular interest. In terms of the notation of this paper, this translates to the parameter value $\tau$ equaling unity, and lead to $x_{12}$ equaling $G$, and the fourth arm of the $2^{\text {nd -iterate thereby being eliminated. More to the point, we observe }}$ its third arm is the identity map. Routine algebra furnishes the following specialization of (1).

$$
\begin{equation*}
f_{r}(x)=-\frac{G x}{G-(G+1) x} \text { and } \frac{d \log f_{r}(x)}{d \log x}=-\frac{f_{r}(x)}{x} \tag{5}
\end{equation*}
$$

This yields the implication that once $x$ enters the absorbing interval $[1, G]$, it alternates between the two points on the right arm of the M-map. Moving on to the specialization of $2^{\text {nd }}$-iterate, we obtain

$$
f^{2}(x)= \begin{cases}((1-\beta) / \beta)^{1+\alpha}(x)^{\alpha^{2}} & \text { if } 0 \leq x \leq(\beta /(1-\beta))^{1 / \alpha}  \tag{6}\\ \frac{(1-\beta) x^{\alpha}}{\beta+x^{\alpha}} & \text { if }(\beta /(1-\beta))^{1 / \alpha} \leq x \leq 1 \\ x & \text { if } 1 \leq x \leq(\beta /(1-\beta))^{-1}\end{cases}
$$

It is the fact that $f_{r}^{2}(x)=x$ and the right arm of the M-map being the square root of the identity over the interval $[1, G]$ that gives rise to a continuum of two-period cycles when $\tau=1$. This is of particular interest in that it allows a complete characterization of all the higher iterates of the M-map ${ }^{15}$

The formulae look formidable, but the basic pattern is clear enough. Over the interval $[1, G]$, the $n^{t h}$ iterate is the identity map for all even integers $n$, and is the restriction of the M-map itself to the same interval for odd $n$. More generally, the remaining interval $[0,1]$, is divided into $n$ intervals with $n$ arms, and the basic observation is that for each subsequent iteration, $f^{-1}$ furnishes only one additional point that results in only the first arm splitting into two. The $5^{\text {th }}$ and $6^{\text {th }}$ iterates pictured in Figure 3 give the necessary intuition. The first of these has 5 arms over the interval [ 0,1 ], and by virtue of (5), a continuum of 2 -period cycles, with the implication that the original map has a 10 -period cycle. Indeed, a continuum of them as the picture of the $10^{\text {th }}$-iterate would establish. More generally, for odd $n$ an $n^{t h}$-iterate exhibiting 2-period cycle has the implication that the $2 n^{\text {th }}$-iterate has a continuum of fixed points, and thereby a continuum of $2 n$ cycles. Note also that all points are periodic or eventually (not asymptotically) periodic. In terms of the language used in [39], the Matsuyama system is not chaotic, and the fact that for the particular value $\tau=1$ under consideration, it is not difficult to demonstrate analytically that all cycles have the period length of a power of 2 .

[^3]We can now present the general case. On defining $x_{n}$ as

$$
f_{l}^{n}\left(x_{n}\right)=1, \text { or equivalently, } x_{n}=G^{-\left(1+\alpha+\cdots+\alpha^{n-1}\right) / \alpha^{n}}, \text { for } n=1,2, \ldots,
$$

we can assert the following.
Proposition 2. For $\tau=1$, and for $k=1,2, \ldots$ and $1 \leq \ell \leq k-1$, the even and odd iterates of the M-map are given by

Proof. In light of the earlier observations, the proof follows easily by induction. First, it is easy to verify that $f^{2 k}(\cdot)$ and $f^{2 k+1}(\cdot)$ are identical to $f^{2}(\cdot)$ and that $f^{3}(\cdot)$ when $k=1$. Using the fact that $f_{r}^{2}(\cdot)$ is an identity map, we can show that $f^{2 k+1}(x)=f\left(f^{2 k}(x)\right)$ and $f^{2(k+1)}(x)=f\left(f^{2 k+1}(x)\right)$.

We can now move beyond the second iterate.

## 3 The Non-existence of a 3-period Cycle

This tripartite section presents the third iterate, comments on the antecedent literature regarding the non-existence claim, and then offers its proof.

### 3.1 The Third Iterate of the M-map

In the determination of the third iterate, we stay with the basic procedure already delineated in the determination of the second iterate. Instead of the vertical through $(1, G)$, as in the determination of the second iterate, we focus on the critical benchmarks $x_{21}, x_{22}$ and $x_{23}$, as shown in Figure 2, and constitutive elements of the set $\left(f^{2}\right)^{-1}$. Unlike the case of the tent-map, for example, we do not obtain four points but only three: the
procedure does not keep doubling the turning points because both arms of the M-map are not "onto." Thus, we obtain the seven intervals that go towards the determination of the third iterate. In any case, the procedure is now transparent; what is of relevance is that one can, in particular, chart out the turning points of as many iterates as we like, all on one map.

We present the algebraic specification of third iterate in the Appendix, and turn the reader's attention to Figure 2. Note that $f^{2}\left(x_{21}\right)=f^{2}\left(x_{22}\right)=f^{2}\left(x_{23}\right)=1$, leading to $f^{3}\left(x_{21}\right)=f^{3}\left(x_{22}\right)=f^{3}\left(x_{23}\right)=G$. We also obtain $f^{3}\left(x_{11}\right)=\tau$ and $f^{3}\left(x_{12}\right)=\tau$. Finally, note that $f^{3}(1)=G \tau^{\alpha}$ and $f^{3}(G)=f\left(G \tau^{\alpha}\right)=\rho$, where the new parameter

$$
\begin{equation*}
\rho \equiv \frac{\beta G^{2} \tau^{\alpha}}{\beta-1+G \tau^{\alpha}} . \tag{7}
\end{equation*}
$$

Remaining with Figure 2, we see that the third iterate intersects the 45 -degree line only at the fixed point $\hat{x}$ of the M-map itself, and the two other fixed points of the second iterate designating a 2 -period cycle, have disappeared. But of course, a rigorous proof of the non-existence of a 3 -period cycle requires argumentation that goes beyond a picture. It requires a proof that the third iterate intersect the 45 -degree line only at one point for all values in the admissible two-dimensional manifold of parameters, and that such a proof has not yet been offered in the antecedent literature since Matsuyama presented his model in 1999.

### 3.2 The Claim and its Antecedent Literature

The following claim of obvious consequence for an understanding of the M-map and its attendant complexity is pervasive in the antecedent literature.

Theorem 1. There does not exist a 3-period cycle in the admissible parameter manifold of the M-map.

In this subsection, we ask whether there exists a formal proof of the above claim in the literature. We have already referred to Mitra's sufficient condition for Li-Yorke chaos, and his taking note of the following seminal footnote in [39].

It is straightforward to show that this system is not chaotic in the sense of Li-York (sic), by demonstrating the non-existence of period-3 cycles. For this it suffices to show that $\Phi^{3}\left(k_{c}\right)>k_{c}, \cdots$ Note, however that this does not rule out the possibility of chaotic trajectories. To rule out such a possibility, one needs to show that all the cycles have period length of a power of 2 , a property that is difficult to demonstrate analytically ${ }^{16}$

[^4]The point is that the system is indeed chaotic in the sense of Li-Yorke, and that the (rather obvious) reason why the terse remarks in the footnote do not constitute a proof is simply that the Li-Yorke theorem requires that there exist a 3-period cycle starting from anywhere, and not necessarily one including the kink. It is this reason that makes Mitra's sufficient condition for Li-Yorke chaos all the more relevant for the M-map. As far as the literature since [39] and [42] is concerned, there is no mention of 3-period cycles in [45] except for a general remark on Li-Yorke chaotic maps; see [45, Footnote 10]. In their section on "chaotic intervals", [19, Section 2.3], the authors write, "Indeed as we see in Figure 2, it is also correct to say that cycles of period three cannot exist." However, in reference to their Figure 2, we read:

The rigorous proof of the bifurcations occurring in the map [is] not easy, because of the complex analytical expressions. However, a numerical proof can first be given ${ }^{17}$

The challenge posed by the M-map is that it is non-linear, and plane geometry can fool the eye: in Figures 7 a and 7 b below, there is clearly a 3 - and 5 -period cycle starting from the critical point. One simply has to choose the parameters $(\beta, G)$ properly, but as we see from Figure 7 , and rigorously from the theorem (and its proof) to follow, such parameters do not exist!

Indeed, since numerical proofs, by necessity, have to approximate smooth segments by piecewise linear ones, we depict a piecewise-linear map in the left panel of Figure 4 . This map has the main qualitative features of the M-map in that the left arm is (weakly) concave and the right arm is convex. Although $f^{3}(1)>1$, and there is no 3-period cycle starting from the critical point, it can be easily seen that there exist two 3-period cycles elsewhere on the map ${ }^{18}$

$$
f(x)=\left\{\begin{array}{cc}
1.1+3.9 x & \text { if } x \leq 1 \\
7.5-2.5 x & \text { if } \bar{x} \geq x>1 \\
\{(2.5 \bar{x}-7.5) /(5-\bar{x})\}(x-5) & \text { if } 5 \geq x \geq \bar{x}
\end{array}\right.
$$

where $\bar{x}=2.95$. The map and its third iterate is depicted in Figure 4 .

### 3.3 A Proof of the Claim

The point then is that a rigorous analytical proof of the non-existence of a 3-period cycle is needed. We provide such a proof in the remainder of this section based on the third iterate of the M-map. Towards this end, we shall also need the inequalities collected as Lemma 1 below; their proof is relegated to the Appendix and they can

[^5]perhaps be profitably skipped by a reader interested only in the qualitative, rather than the quantitative, aspects of the M-map.

Lemma 1. For any $G>1$, the following inequalities hold:

$$
\left(\frac{(1-\beta)^{2}-\beta^{2} G^{3}}{(1-\beta-\beta G) G}\right)^{1 / \alpha}<x_{11}<\tau
$$

We can now turn to the proof of Theorem 2.
Proof. We begin the proof with the assertion that the interval $[\tau, G]$ constitutes an absorbing interval. Formally, we want to show that for any $x>0$, there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n>N, f^{n}(x) \in[\tau, G]$. First, $f([\tau, G])=[\tau, G]$, so if $x \in[\tau, G]$, then $f^{n}(x) \in[\tau, G]$ for any $n \in \mathbb{N}$. Second, consider $x \in(0, \tau)$. Note that $G x^{\alpha}>G x$. Pick $N$ to be the smallest integer that is greater than $(\ln \tau-\ln x) / \ln G$. Then $f_{l}^{N}(x)>\tau$, so $f^{n}(x) \in[\tau, G]$ for any $n \in \mathbb{N}$ with $n>N$. Last, the argument concerning $x \in(0, \tau)$ carries through for $x>G$ because $f(x) \in(0, \tau)$.

Next, we assert that a 3-period cycle, if it exists, must be on the absorbing interval $[\tau, G]$. First, at least one periodic point is on the left arm of the M-map. If this is not the case, then according to the $3^{r d}$ iterate of the M-map (also see the $6^{\text {th }}$ arm in Figure 2), the following equation must admit at least three distinct roots:

$$
x=\frac{\beta^{3} G^{3} x}{(\beta-1)^{3}+\left[(\beta-1)^{2}+(\beta-1) \beta G+\beta^{2} G^{2}\right] x} .
$$

There is a unique solution to the equation above, which is the fixed point of $f$, so a 3 -period cycle cannot occur entirely on the right arm of the M-map. Second, at most one point of a 3-period cycle must be on the left arm of the M-map. If this is not the case, two periodic points on the left arm must occur consecutively in a 3-period cycle. However, since we know $G \tau^{\alpha}>1$, which implies $G x^{\alpha}>1$ for any $x \in[\tau, 1]$, this leads to a contradiction.

We have now shown that there must be one point on the left arm and two points on the right arm in a three-period cycle. According to our formula of the third iterate of the M-map (also see the $3^{r d}$ arm in Figure 2), this suggests that there exists $\hat{x} \in[\tau, 1]$,

$$
\frac{\beta^{2} G^{3} \hat{x}^{\alpha}}{(\beta-1)^{2}+(\beta-1+\beta G) G \hat{x}^{\alpha}}=\hat{x} .
$$

Since

$$
\frac{\beta^{2} G^{3} x^{\alpha}}{(\beta-1)^{2}+(\beta-1+\beta G) G x^{\alpha}}>x^{\alpha} \text { iff } x>\left(\frac{(1-\beta)^{2}-\beta^{2} G^{3}}{(1-\beta-\beta G) G}\right)^{1 / \alpha}
$$

according to Lemma 1, this implies

$$
\frac{\beta^{2} G^{3} x^{\alpha}}{(\beta-1)^{2}+(\beta-1+\beta G) G x^{\alpha}}>x^{\alpha} \text { for } x \geq \tau
$$

which further implies that $\hat{x}>\hat{x}^{\alpha}$. However, $x^{\alpha}>x$ for $x<1$, and we obtain the contradiction and complete the proof.

It bears emphasis that two of the steps used in the proof presented above are available in the literature. First, the claim that the interval $[\tau, G]$ is an absorbing interval constitutes Lemma 6.1 in [61]. ${ }^{19}$ Second, the proof relies on Lemma 1 whose proof is relegated to the Appendix, and where we explicitly note that only one of the inequalities of the lemma is available as Footnote 8 in [34]. However, the proof presented above relies on both inequalities.

Next, we move on to the fourth iterate, and to the possibility of a 5 -period cycle.

## 4 The Non-existence of a 5-period Cycle

The fact that the M-map is not complicated enough to have a 3-period cycle anywhere in its manifold of admissable parameters is a folk-result in the sense that it is well-known, even if it was not so far proved. What is not well-known, and to the knowledge of the authors never asked, is whether the M-map is complicated enough to have a 5 -period cycle anywhere in its manifold of admissable parameters. We turn to this question in this section.

### 4.1 The Fourth Iterate of the M-map

Consider Figure 5 and the fourth iterate of the M-map. As mentioned in the opening paragraph of Section 3, the procedure to determine it is by now quite routine. Instead of the set $\left(f^{2}\right)^{-1}(1)$, we work with $\left(f^{3}\right)^{-1}(1)$ to obtain the five points $x_{3 i}, i=1, \cdots, 5$, and, in general, the twelve intervals depicted in Figure 5. The qualifier "in general" highlights the fact that there may be only 11 arms to the fourth iterate depending on the parameter values. All this is a continuing testament to the fact that unlike the tent-map, one of the two arms of the M-map is not "onto" the unit interval. ${ }^{20}$

Again, we spare the reader the detailed algebraic specification of the fourth iterate by relegating it to the Appendix, and focusing his/her attention on its diagrammatic representation in Figure 5. We see that the fourth iterate intersects the 45-degree line at

[^6]five points: the fixed point $\hat{x}$ of the M-map itself, the unique 2-period cycle and another unique 4-period cycle. These findings can be usefully compared with Figures 4 and 5 in [19]. ${ }^{21}$ In looking at Figure 5, note that in line with the iterative procedure we are following, $f^{3}\left(x_{3 i}\right)=1, i=1, \cdots 5$, necessarily implies $f^{4}\left(x_{3 i}\right)=G, i=1, \cdots 5$. Note also that
$$
f^{4}\left(x_{21}\right)=f^{4}\left(x_{22}\right)=f^{2}\left(x_{23}\right)=\tau \text { and } f^{4}\left(x_{11}\right)=f^{4}\left(x_{12}\right)=G \tau^{\alpha}
$$
that $f^{4}(1)=\rho$ and that $f^{4}(G)=f(\rho)$. Finally, note that there exists the $12^{\text {th }}$ arm if and only if $G>x_{35}$, or equivalently, $\tau<x_{22}$, which is also equivalent to $\rho>1$, and that the $13^{\text {th }}$ arm does not exist because we have already shown that $\tau>x_{11}$.

### 4.2 The Result and its Proof

The surprise here is that to show the non-existence of a 5-period cycle, we do not need to compute the fifth iterate and to show that it intersects the 45 -degree line only at one point for all values in the admissible two-dimensional manifold of parameters. The algebraic specification of the fourth iterate suffices! We begin with a statement of the second principal claim of the paper.

Theorem 2. There does not exist a 5-period cycle in the admissible parameter manifold of the M-map.

The proof of the result relies on the following inequalities, and we again relegate their proofs to the Appendix.

Lemma 2. For any $G>1$, the following inequalities hold:

$$
\text { (i) } G \tau^{\alpha}>x_{23}, \text { (ii) } \rho<x_{12}, \text { (iii) }\left(\frac{(\beta-1)^{4}-\beta^{4} G^{5}}{G(1-\beta-\beta G)\left[(\beta-1)^{2}+\beta^{2} G^{2}\right]}\right)^{1 / \alpha}<x_{33}
$$

We can now turn to the proof of the Theorem.
Proof. First, a 5-period cycle can only occur over the range $[\tau, G]$. According to Lemma 1. this implies that a fixed point of $f^{5}(\cdot)$ must be no less than $x_{11}$. Second, if there is a five-period cycle, at least one point on the cycle must be on the left arm. This can be seen by solving the fixed point for the fifth-iterate of the right arm given by

$$
\frac{\beta^{5} G^{5} x}{(\beta-1)^{5}+\left[(\beta-1)^{4}+\beta G(\beta-1)^{3}+\beta^{2} G^{2}(\beta-1)^{2}+\beta^{3} G^{3}(\beta-1)+\beta^{4} G^{4}\right] x}=x .
$$

[^7]There is a unique fixed point and it coincides with the fixed point of $f(\cdot)$, so it is impossible to have a five-period cycle with all the points on the right arm of the original M-map. Hence, for there to be a five-period cycle, there must exist a fixed point of $f^{5}(\cdot)$ in $\left[x_{11}, 1\right]$.

We first consider the interval $\left[x_{11}, x_{33}\right]$. The fourth iterate on this interval increases with $x$, and we have $f^{4}\left(x_{11}\right)=G \tau^{\alpha}>1$ and $f^{4}\left(x_{33}\right)=G$. This implies that the fifth iterate on this interval decreases with $x$. According to Lemma 2 (i), $G \tau^{\alpha}>x_{23}$, so $f^{5}\left(x_{33}\right)=\tau>\left(x_{23} / G\right)^{1 / \alpha}=x_{33}$. Therefore, $f^{5}(x) \geq f^{5}\left(x_{33}\right)>x_{33} \geq x$ for $x \in\left[x_{11}, x_{33}\right]$. There is no fixed point of the fifth-iterate on this interval.

Next, consider the interval $\left[x_{33}, x_{22}\right]$. The fourth iterate on this interval decreases with $x$, and we have $f^{4}\left(x_{33}\right)=G$ and $f^{4}\left(x_{22}\right)=\tau<1$. Define $x_{4} \in\left[x_{33}, x_{22}\right]$ such that $f^{4}\left(x_{4}\right)=1$. Then the fifth iterate increases with $x$ for $x \in\left[x_{33}, x_{4}\right]$. It can be shown that

$$
\begin{gathered}
\frac{\beta^{4} G^{5} x^{\alpha}}{(\beta-1)^{4}+\left[(\beta-1)^{3}+\beta G(\beta-1)^{2}+\beta^{2} G^{2}(\beta-1)+\beta^{3} G^{3}\right] G x^{\alpha}}>x^{\alpha} \\
\Longleftrightarrow x>\left(\frac{(\beta-1)^{4}-\beta^{4} G^{5}}{G(1-\beta-\beta G)\left[(\beta-1)^{2}+\beta^{2} G^{2}\right]}\right)^{1 / \alpha}
\end{gathered}
$$

According to Lemma 2(iii), this implies the inequality holds for $x \geq x_{33}$. Moreover, $x^{\alpha}>x$ for $x \leq x_{4}<1$. Combining these two inequalities, we have $f^{5}(x)>x$ for $x \in\left[x_{33}, x_{4}\right]$. The fifth iterate decreases with $x$ for $x \in\left[x_{4}, x_{22}\right]$. We have $f^{5}(x) \geq$ $f^{5}\left(x_{22}\right)=G \tau^{\alpha}>1>x_{22} \geq x$. Therefore, there is no fixed point in $\left[x_{33}, x_{22}\right]$.

Consider the last interval $\left[x_{22}, 1\right]$. The fourth iterate on this interval increases with $x$. There are two possible cases. If $f^{4}(1)=\rho \leq 1$, the fifth arm increases with $x$ on this interval, so $f^{5}(x) \geq f^{5}\left(x_{22}\right)>1 \geq x$. If $f^{4}(1)>1$, the fifth arm first increases and then decreases with $x$ for $x \in\left[x_{22}, 1\right]$. However, as Lemma 2 (ii) implies $\rho<(1-\beta) /(1-G \beta)$ or equivalently $f(\rho)=(G \beta \rho) /(\beta-1+\rho)>1$, we again have $f^{5}(x)>1 \geq x$ for any $x \in\left[x_{22}, 1\right]$. There is no fixed point in $\left[x_{22}, 1\right]$ and the proof is complete.

We invite the reader to compare the global figure, Figure 5, with the corresponding local and numerical pictures presented in Figures 4 and 5 in [19] and Figure 2 in [45].

## 5 The Existence of Cycles of 6- and Higher Periods

As emphasized in the introduction, the motivation of this work is to shift the emphasis from numerical determinations of the existence of "chaos" to an analytical examination of how rich and complicated are the dynamics that one can associate with the M-map. We have also been emphasizing that our primary concern is with questions of existence rather than that of the stability or the genericity of periodic orbits. As such, we ask for the smallest value of odd $n>5$ for which $n$-period cycles, stable or unstable, exist?

### 5.1 A 7-period Cycle

A numerical answer is furnished in the following example that presents a trajectory of a 7-period cycle that includes the critical point (kink) of the M-map.

Example 1. For $\alpha=0.99$, there exists $G^{*} \in(1,(1-\beta) / \beta)\left(G^{*} \approx 1.01016\right)$ such that $a$ 7 -period cycle starts from the critical point of the M-map.

The example is diagrammed in Figure 6. It is also of some interest that we can underscore our analytical findings regarding the fourth iterate by an example within the same parametric regime given by $\alpha=0.99$.

Example 2. For $\alpha=0.99$, there exists $\left.G^{* *} \in(1,(1-\beta) / \beta)\left(G^{* *} \approx 1.29886\right)\right)$ such that a 4-period cycle starts from the critical point of the M-map.

These examples, and especially the first, are of interest in their own right, but we now use them instrumentally to present a result in the following subsection: it completes the existence question that we pose in this paper, and can be usefully compared with Claim 12 and Proposition 1 in [45]. Let us also take this opportunity to draw attention to the fact that Sharkovsky's theorem guarantees that in this parametric regime, there exist cycles of all periods greater than or equal to seven. ${ }^{22}$

### 5.2 A General Result

In [30], the authors show that the so-called check-map associated with the so-called RSS mode ${ }^{233}$ has the versatility that under its various parameter values, it can exhibit topological chaos and $n$-period cycles for any $n \geq 1$. Its "anything-goes-construction", herein referred to as the KP-construction, relies on pull-back trajectories for points starting at the critical point (kink) of the check-map.$^{24}$ It is a natural to ask whether that geometry can be transcribed to the M-map: does the KP-construction work for the M-map? While referring to [30] for details, we shall answer this question through Figures 7 a to 7 d that implement the analogue to the KP-construction for the M-map.

Towards this end, return to Figure 1, and to the absorbing interval $(\tau, G)$ used in the proof of Theorem 2 above. Once a trajectory enters this interval, it remains within it. ${ }^{25}$ More to the point, the restriction of the M-map to this interval transforms it by

[^8]interchanging the characteristic signatures of its two arms: its right arm is now "onto", and its left arm ends in the interior of the left half of the absorbing square. We can now simply follow the geometry of [30], and given any $n$-period trajectory that we desire, mark off the relevant point on the left hand side of the square to determine where the left arm should end. Figures 7 a and 7 b present examples of M-maps exhibiting 3- and 5period cycles derived in this manner. Since we have analytically proved the non-existence of a 3-period and a 5-period cycle in the M-map, something is obviously wrong! And it is this dissonance that furnishes our first observation: like the check-map, the M-map is also a two-parameter map, but unlike the former, it has the simultaneity under which a specification of one parameter limits the range of the other parameter. In short, the map exhibited in Figures 7 a and 7 b is not a legitimate M-map (see Figure 7 c ), but it is one in Figure 7d. Put differently, and more technically, the two-dimensional parametric manifold of the M-map is not a rectangle but a more complicated object that merits, indeed demands, further scrutiny and investigation.

Consider an $n$-period cycle with its orbit specified in the following way: it starts from the critical point 1, and, after hitting $G$ and $G \tau^{\alpha}$, it stays on the right arm of the M-map until it hits the critical point again. Figure 6 illustrates an example of this type of cycle. Formally, there exists an $n$-period $(n \geq 3)$ cycle with this particular orbit if $f_{r}^{n-3}\left[f_{l}\left(f_{r}(1)\right)\right]=1$, or equivalently,

$$
G \tau^{\alpha}=\left[\frac{1}{1+G \beta-\beta}+\left(\frac{G \beta}{\beta-1}\right)^{n-4} \frac{G \beta^{2}(1-G)}{(1-\beta)(1+G \beta-\beta)}\right]^{-1} \equiv \varphi_{n}(\beta, G)
$$

We can now consider a subset of the 2-dimensional parameter space given by

$$
\mathcal{P}_{n}=\left\{(\beta, G) \in(1 / e, 1 / 2) \times(1,(1-\beta) / \beta): G \tau^{\alpha}=\varphi_{n}(\beta, G)\right\}
$$

Then Theorems 2 and 3 imply that $\mathcal{P}_{n}=\emptyset$ for $n=3,5$. The question then pertains to higher values of $n$. We can use Examples 1 and 2 above to present the following consequence of the intermediate value theorem.

Proposition 3. For $n \geq 3, \mathcal{P}_{n} \neq \emptyset$ for all $n \neq 3$ and 5.
Proof. For $G \in(1,(1-\beta) / \beta)), \varphi_{n}(\beta, G) \equiv y_{n} \in\left(y_{7}, y_{4}\right)$ for any $n \geq 6$ and $n \neq 7$. In Examples 1 and 2, we have shown that for $\alpha=0.99, G^{*} \tau^{\alpha}=y_{7}$ and $G^{* *} \tau^{\alpha}=y_{4}$. Therefore, when $\alpha=0.99, n \geq 6$, and $n \neq 7$, we have $y_{n}>y_{7}=G \tau^{\alpha}$ for $G=G^{*}$ and $y_{n}<y_{4}=G \tau^{\alpha}$ for $G=G^{* *}$. When $\alpha$ is given, both $G \tau^{\alpha}$ and $y_{n}$ are continuous in $G$, so according to the intermediate value theorem, there exists $G \in\left(G^{* *}, G^{*}\right)$ such that $y_{n}=G \tau^{\alpha}$ for $n \geq 6$ and $n \neq 7$. According to Theorems 1 and 2, the condition above cannot hold for $n=3$ or 5 . This completes our proof.

However useful numerical examples prove to be, they surely need underscoring by an analytical result. A natural question concerns parameter values under which 7-period
cycles exists! We answer this question by appealing to the generalization of the Li-Yorke theorem presented in Li-Misurewicz-Piangiani-Yorke [35], henceforth LMPY, and further refined by Ruette [51, Proposition 3.34]. We can present
Theorem 3. A sufficient condition for the existence of a 7-period cycle of the M-map is given by

$$
f^{7}(1)=f^{3}(\rho)=\frac{\beta^{4} G^{5} \tau^{\alpha}}{(\beta-1)^{4}+\left[(\beta-1)^{3}+\beta G(\beta-1)^{2}+\beta^{2} G^{2}(\beta-1)+\beta^{3} G^{3}\right] G \tau^{\alpha}} \leq 1
$$

Proof. According to the LMPY theorem, there exists a 7-period cycle if $f^{7}(x) \leq x<f(x)$ for some $x$. In applying this result, we work with $x=1$. Since $f(1)=G>1$, for a 7 -period cycle, we need conditions that guarantee $f^{7}(1) \leq 1$. We have already shown that $f^{2}(1)=\tau<1$ and that $f^{3}(1)=G \tau^{\alpha}>1$. The complication arises from the fact that we require

$$
f^{4}(1)=\frac{\beta G^{2} \tau^{\alpha}}{\beta-1+G \tau^{\alpha}}=\rho>1
$$

when in general it can take values either greater or less than one. We can now assert that $f^{5}(1)>1$. If not, we would have a 5 -period cycle, and a contradiction to Theorem 2 above. We further require $f^{6}(1) \geq 1$. This implies that

$$
f^{7}(1)=\frac{\beta^{4} G^{5} \tau^{\alpha}}{(\beta-1)^{4}+\left[(\beta-1)^{3}+\beta G(\beta-1)^{2}+\beta^{2} G^{2}(\beta-1)+\beta^{3} G^{3}\right] G \tau^{\alpha}}
$$

Thus, as a consolidation of the argumentation, in order to show the existence of a 7-period cycle, we need to show (i) $f^{4}(1) \geq 1$, (ii) $f^{6}(1) \geq 1$, and (iii) $f^{7}(1) \leq 1$.

Towards this end, we first show (iii) implies (i) and (ii). If $f^{6}(1) \in[\tau, 1$ ), then $f^{7}(1)>1$, and we contradict (iii). Now suppose $f^{4}(1)<1$. This implies $G \tau^{\alpha}>$ $(1-\beta) /(1-\beta G)$. Furthermore, we are working under a hypothesis that implies that

$$
G \tau^{\alpha} \leq \frac{(\beta-1)^{4}}{\beta^{4} G^{4}-(\beta-1)^{3}-(\beta-1)^{2} \beta G-(\beta-1) \beta^{2} G^{2}-\beta^{3} G^{3}}
$$

These two inequalities suggest that

$$
\begin{gathered}
\frac{1-\beta}{1-\beta G}<\frac{(\beta-1)^{4}}{\beta^{4} G^{4}-(\beta-1)^{3}-(\beta-1)^{2} \beta G-(\beta-1) \beta^{2} G^{2}-\beta^{3} G^{3}} \\
\quad \Longleftrightarrow z(G) \equiv \beta^{2} G^{3}-\beta G^{2}+(1-\beta) G-(1-\beta)^{2}<0
\end{gathered}
$$

Now observe that $z(1)=0$, and $z^{\prime}(G)=3 \beta^{2} G^{2}-2 \beta G+(1-\beta)$. We can show that $z^{\prime}(1)>0$ and that $z^{\prime \prime}(G)>0$. Thus, $z^{\prime}(G)>0$ and that therefore $z(G)>0$ for any $G>1$. We have obtained the contradiction that we seek, and completed the proof.

We now conclude this section with
Example 3. For $\alpha=0.99$, there exists $G=1.01000$ such that the inequality identified in Theorem 3 holds strictly.

## 6 Topological Entropy and Types of Topological Chaos

In this section, we return to the theme of what constitutes chaos and a chaotic dynamical system? More specifically, what does the existence of a 7-period cycle tell us about the rich complexity of the dynamics of the M-map?

We begin with the following definition ${ }^{26}$
Definition 1. For any $n \in \mathbb{N} \cup 2^{\infty}$, a continuous mapping $f$ from the unit interval to itself, an interval map, is of type $n$ for Sharkovsky's order if the periods of the periodic points of $f$ form exactly the set $\{m \in \mathbb{N}: m \unrhd n\}$, where the notation $\left\{m \in \mathbb{N}: m \unrhd 2^{\infty}\right\}$, stands for $\left\{2^{k}: k \geq 0\right\}$, and $\unrhd$ for Sharkovsky's order with equality ${ }^{27}$

We can now state that the M-map, as specified in Equation (2) without any additional restrictions on its parameters, is of Sharkovsky's type 7. This translates into further numerical estimates as a consequence of a result ascribed to Block-Guckenheimer-MisiurewiczYoung, henceforth BGMY, and reported in [51, Theorem 4.60].

Theorem 4 (BGMY). If an interval map $f$ has a 7-period cycle, then its topological entropy is bounded below by $\log \lambda_{7}$ where $\lambda_{7}$ is the unique positive root of $x^{7}-2 x^{5}-$ 1. Moreover, there exists an interval map with a periodic point of period 7 and whose topological entropy is equal to $\log \lambda_{7}$.

It is easy to calculate the unique positive root of the polynomial in Theorem 4 to be 1.4656. This numerical viewpoint allows one to see Theorems 1 and 2 above as assertions that the topological entropy of the M-map, for any point in its admissable manifold of parameters, cannot be pushed up to the unique positive root of the polynomial corresponding to a 3 -period cycle $x^{3}-2 x-1$, and thereby to $(1+\sqrt{5}) / 2 \approx 1.6180$, or even to the unique positive root of the polynomial corresponding to a 5 -period cycle $x^{5}-2 x^{3}-1$, and thereby to 1.5129 . The question is whether we can go beyond Li-Yorke chaos, and use the existence of a 7 -period cycle and a positive lower bound for the topological entropy for the M-map, to say more? To return to the 7 -fold criteria with which we began the introduction, what is Li-Yorke chaos anyway? Why is it to be privileged? Does it exhaust the meaning that can be given to chaos, even topological chaos?

In his pioneering contribution of more than fifteen years ago, Mitra had already shifted the focus to turbulence, and had referred to the Li-Yorke criterion, rather than to Li-Yorke chaos ${ }^{28}$ Using [42, Section 2] as a foothold ${ }^{29}$ we turn to the epigraph of this

[^9]essay, and to the conceptions articulated as (a) to (c) of its introduction. Blanchard [4, p. 25] distinguishes five different types of topological chaos: "Li and Yorke's, Auslander and Yorke's, Devaney's, sensitivity and positive entropy." In [5], the focus is on Li-Yorke pairs, and in [6], on the size of scrambled sets ${ }^{30}$ The tripartite nature of chaos in the sense of Devaney - its transitivity, density of periodic points and sensitivity to initial conditions - allows additional notions by differing permutations of those conditions. ${ }^{31}$ Thus, Ruette [51, Remarks 7.5 and 7.12] writes:

A topological dynamical system $(X ; f)$ is sometimes called chaotic in the sense of Auslander-Yorke or chaotic in the sense of Ruelle and Takens if it is transitive and sensitive to initial conditions, and chaotic in the sense of Wiggins if there exists an invariant closed set $Y \subset X$ such that $\left(Y ;\left.f\right|_{Y}\right)$ is transitive and sensitive to initial conditions, and strongly chaotic in the sense of Xiong if $Y$ is not necessarily closed, and $f$ is topologically mixing, the set of periodic points is dense in $Y$ and their periods form an infinite set.

For the possibly disoriented reader, we offer the following orientation. It is a portmanteau theorem that deserves circulation among economists interested in dynamical systems.

Theorem 5. For an interval map $f$, the following assertions are equivalent:
(i) the topological entropy of $f$ is positive,
(ii) $f$ has a periodic point whose period is not a power of 2,
(iii) there exists an integer $n \geq 1$ such that $f^{n}$ has a strict horseshoe,
(iv) $f$ has a homoclinic point,
(v) $f$ has an eventually homoclinic point,
(vi) there exists an invariant closed set $X$ such that $\left(X ;\left.f\right|_{X}\right)$ is chaotic in the sense of Devaney,
(vii) there exists an infinite invariant closed set $X$ such that $\left(X ;\left.f\right|_{X}\right)$ is transitive and $X$ contains a periodic point,
(viii) there exists a positive integer $n$ and an uncountable $f^{n}$-invariant set $X$ such that $\left(X ; f^{n} \mid X\right)$ is topologically mixing and the set of periodic points is dense in $X$,
(ix) there exists a positive integer $n$ and an $f^{n}$-invariant set $X$ such that $\left(X ; f^{n} \mid X\right)$ is topologically mixing,
(x) there exists a positive integer $n$ and an infinite $f^{n}$-invariant set $X$ such that $\left(X ; f^{n} \mid X\right)$ is transitive,
(xi) $f$ exhibits distributional chaos of any of the three types, $D C 1, D C 2$ and $D C 3$.

[^10]Proof. Since the theorem is simply a consolidation of results collected and dispersed in [51], we present a key to its proof for the interested reader in a way that brings out the historical evolution of the ideas. This is to say, with names and dates.

For the equivalence of (i) to (ii)-(iii), see Theorem 4.58 in [51] where the equivalence of (i) and (ii) is referred to as "one of the most striking results in interval dynamics." ${ }^{32}$ For the equivalence of (i) and (vi) to (vii), see Theorem 7.3 in [51] where it is ascribed to a 1993 paper of Shihai Li. For the equivalence of (i) and (viii) to (x), see Corollary 7.9 [51] where it is deduced from two results that are coupled with the equivalence of (i) and (ii) above, and ascribed to an unpublished 1988 paper of Jin-Cheng Xiong. The equivalence of (i) to (viii) is ascribed to a 1981 paper of Osikawa-Oono. For the equivalence of (i) to (xi), see Corollary 6.27 in [51] where it is deduced from two results ascribed to a 1994 paper of Schweizer and Smítal.

Theorem5is conceptually and terminologically heavy. While a comprehensive exposition is well beyond the scope of this essay ${ }^{33}$ we limit ourselves to seven observations pertaining to it. First, a Li-Yorke pair is sufficient to characterize Li-Yorke chaos, and that either is a consequence of positive topological entropy ${ }_{4}^{34}$ Second, the theorem avoids the term turbulence in favor of Misiurewicz' modification of Smale's notion of a horseshoe for twodimensional smooth homeomorphisms: $\left(J_{1}, J_{2}\right)$ constitute a strict horseshoe if they are closed non-degenerate disjoint intervals such that $J_{1} \cup J_{2} \subset f\left(J_{i}\right)$ for $i=1,2$. Third, a homoclinic point is one whose forward orbit "hits" a periodic point in a finite number of periods, and whose backward orbit converges to it asymptotically ${ }^{35}$ Fourth, a transitive sensitive subsystem, riding on the old and essential ideas of transitivity and sensitivity, is strictly intermediate between positive entropy and Li-Yorke chaos. Fifth, the former strictly implies the latter ${ }^{36}$ Sixth, topological sequence entropy, though not as old as transitivity, characterizes Li-Yorke chaos ${ }^{37}$ Finally, the existence of a periodic point of odd period greater than 1 implies (strong) chaos in the sense of Xiong, and is equivalent to the transitivity of $f^{2}$ on an infinite, $f$-invariant closed subset ${ }^{38}$

[^11]We conclude this section with the assertion that Theorem 5, and the paragraph following it, is simply an underscoring of the exciting developments current in topological chaos ${ }^{39}$

## 7 Ergodic Chaos and Experimental Mathematics

Our approach so far has been determinedly topological and determinedly analytical, and thereby ignores recent work on the M-map that has moved in the direction of ergodic chaos and comprehensive numerical experimentation. For completeness, we take a brief critical note of this work.

Writing in 2011 that "in the existing literature on the Matsuyama model, the extent of non-linearity has not been fully examined," Yano et al. [61] appeal to a 1975 theorem they attribute to Lasota-Yorke-Kowalski-Li (LYKL) to identify a range of parameters for which the M-map exhibits ergodic chaos, precisely defined. They see their result "confirming Mukherji's bifurcation analysis, which establishes a similar result numerically." ${ }^{40}$ In [62], they offer an enlargement of the parametric region identified earlier on the basis of a notion of iterative expansivity which enables a generalization of the LYKL theorem; see [52, 53]. This interesting work can also be profitably read through (i) the statistical viewpoint taken in [9], (ii) recent advances [13, 17, 21] that move the subject across the very boundary of topological and statistical chaos, and (iii) epistemological considerations in [59].

Taking the numerical-analytical distinction as a background, we note the 9-point representation of the dialogue between man and machine presented in [8], part of a multi-volume reflection on experimental mathematics. The authors emphasize the role of the computer in determining conjectures that are "worth a full-fledged attempt at formal proof," but also the importance of "confirming conjectures by rigorous proof." These observations have some direct relevance to the several papers of Gardini et al. [3, 19, 41, 57], and in the context of the M-map, to the findings in [19] of "growth through chaotic intervals" and of the border-collision bifurcation. Both initiate important directions for further investigation, and as such are of seminal rather than definitive importance.

The first of these pertains to the discussion of 4 -period cycles in [19. The authors criticize [45] and contest that "transition to chaos may occur via the standard perioddoubling bifurcation sequence, writing about a stable 4-cycle found after the stabilization of the 4 -cycle, which we show not possible (sic). The only stable cycles of the model are fixed points and 2-cycles." This is established through a bifurcation diagram, as well

[^12]as local and numerical snapshots of the second and fourth iterates ${ }^{41}$ This dissonance between [45] and [19] remains an open question precisely because the non-linearities of the M-map render explicit numerical values of the variety of bifurcations difficult, and especially when the numerical procedures are not made explicit. ${ }^{42}$ Some additional elaboration of this point may be warranted.

It is well-understood that bifurcations lead to abrupt changes in the stylized representation of an iterate and thereby underscore the importance of analytical formulae pining it down globally in the admissible parameter manifold; see for example [43, 44] for early statements. Thus, a comparison of Figures 3 with Figure 2 show the disappearance of the $4^{\text {th }}$ arm of the $2^{\text {nd }}$-iterate, and in the discussion of the $4^{\text {th }}$-iterate at the end of Section 4.2, the emergence and disappearance of the $12^{\text {th }}$ arm is noted. The point is that five bifurcations are identified in [19]: (i) $G=1$, (ii) $\tau=f_{r}^{2}(1)=1$, (iii) $\rho=f_{r} \circ f_{\ell} \circ f_{r}^{2}=1$, (iv) $f_{r}(\rho)=x_{R}$ and (v) $G \tau^{\alpha}=\hat{x}$, where $\hat{x}$ is the unstable fixed point $1+\beta(G-1)$, and $x_{R}$ is the higher value of the unstable 2-period cycle. The open question is whether this list is complete: specifically, whether the parametric equation $\left|f^{\prime}\left(x_{L}\right) f^{\prime}\left(x_{R}\right)\right|=1$ does, or does not, represent another bifurcation to those listed above ${ }^{433}$ This goes to the heart of the difference in the claims of [45] and [19], and given the pronounced non-linearities of the M-map, can hardly be pinned down solely through numerical experimentation ${ }^{44}$

We conclude this section with two final observations concerning [19]. The first concerns the general and vexatious issue of obtaining results for piecewise smooth maps by approximating them with smooth maps ${ }^{45}$ In particular, this calls for validation of the application of Proposition 2 in [19], proved only for smooth maps ${ }^{46}$ to the M-map. The second observation concerns the border-collision bifurcation at $G=1$ discussed in [19, Section 3]. This bifurcation is established by linearizing a piece-wise continuous map at the point of bifurcation ${ }^{47}$ and it has been so applied to the M-map. This application points to an interesting dialectic in that,

[^13]despite the differences in their parametric manifolds, the M-map and the check-map share interesting similarities, and that one can be understood better by focusing on the other 48

## 8 Concluding Methodological Remark

The points this paper makes, and the results it reports, are sharp enough that one need not go beyond the abstract and the introduction so as to summarize them yet again. Perhaps the only methodological point that warrants some additional explication is (g) of the introduction.

The epistemological point is a simple one. Since May's influential 1976 piece in Nature emphasizing complicated dynamics from simple models, there has been a numerical turn in that questions of comparative economic dynamics, and therefore of economic dynamics, depend essentially of identification of parametric ranges within which the map does not change abruptly. This is not true for the earlier literature, both in growth and development as well as in macroeconomics, in which broad qualitative assumptions such as convexity or supermodularity sufficed. Iterates are now sure to play an important and obvious part in further elaborations that discipline ad hoc savings function by optimizing behavior, as for example, in [40, 14, 27, 56, and by adducing additional stylized facts and temporal specifications stemming from short- and the long-run. Much remains to be done.

## 9 Appendix

We begin with the proofs of the lemmata.
Proof of Lemma 1. The second assertion has been proved in [39], and we transcribe the proof there under our notation here. Define $h_{1}(G) \equiv \beta G^{2+1 / \alpha}-(G+\beta-1)$, and note that $h_{1}(1)=0$, and $h_{1}^{\prime}(G)=(2+1 / \alpha) \beta G^{1+1 / \alpha}-1>0$ for $G>1$, because $2+1 / \alpha>3$ and $\beta>1 / e>1 / 3$. This implies $h_{1}(G)>0$ for $G>1$, and so $\tau-x_{11}=h_{1}(G) /\left[G^{1 / \alpha}(G+\beta-1)\right]>0$ for any $G>1$.

For the first assertion, it suffices to show that

$$
\frac{(1-\beta)^{2}-\beta^{2} G^{3}}{(1-\beta-\beta G)}<1 \Longleftrightarrow \beta G-\beta^{2} G^{3}-\beta+\beta^{2}<0
$$

Define $h_{2}(G)=\beta G-\beta^{2} G^{3}-\beta+\beta^{2}$, and note that $h_{2}(1)=0$ and $h_{2}^{\prime}(G)=\beta-3 \beta^{2} G^{2}<0$ for $G>1$. Hence, $h_{2}(G)<0$ for any $G>1$.

Proof of Lemma 2. Since $\tau<1$, it suffices to show for (i) that
$G \tau=\frac{\beta G^{3}}{\beta-1+G}>\frac{(1-\beta)^{2}}{G^{2} \beta^{2}+(1-\beta-\beta G)} \Longleftrightarrow \beta G^{3}\left(\beta^{2} G^{2}+1-\beta-\beta G\right)>(1-\beta)^{2}(\beta-1+G)$.

[^14]Define $h_{3}(G) \equiv \beta G^{3}\left(\beta^{2} G^{2}+1-\beta-\beta G\right)-(1-\beta)^{2}(\beta-1+G)$. We want to show $h_{3}(G)>0$ for $G>1$. Since $h_{3}(1)=0$, we just need to show

$$
h_{3}^{\prime}(G)=5 \beta^{3} G^{4}-4 \beta^{2} G^{3}+3(1-\beta) \beta G^{2}-(1-\beta)^{2}>0 \text { for } G>1 .
$$

Define $\ell(\beta) \equiv h_{3}^{\prime}(1)=5 \beta^{3}-4 \beta^{2}+3(1-\beta) \beta-(1-\beta)^{2}$ Simple calculation shows $\ell(1 / e) \approx$ $0.006>0$, and $\ell^{\prime}(\beta)=15 \beta^{2}-16 \beta+5>0$ for $\beta \in(1 / e, 1 / 2)$, so $h_{3}^{\prime}(1)>0$. Consider the second derivative $h_{3}^{\prime \prime}(G)=20 \beta^{3} G^{3}-12 \beta^{2} G^{2}+6(1-\beta) \beta G=2 \beta G\left[10 \beta^{2} G^{2}-6 \beta G+3(1-\beta)\right]$. It is straightforward to show that $h_{4}(G) \equiv 10 \beta^{2} G^{2}-6 \beta G+3(1-\beta)$ is an increasing function of $G$ and $h_{4}(1)>0$, so $h_{3}^{\prime \prime}(G)>0$. This implies $h_{3}^{\prime}(G)>0$ for $G>1$.

For (ii), we want to show explicitly that

$$
\tau^{\alpha}>\frac{(1-\beta)^{2}}{(1-\beta) G-(1-\beta G) \beta G^{2}} \text { or that } \tau>\frac{(1-\beta)^{2}}{(1-\beta) G-(1-\beta G) \beta G^{2}}
$$

since $\tau<1$. This reduces to the case to (i) above.
For (iii), rearranging the inequality, we want to show

$$
\frac{(\beta-1)^{4}-\beta^{4} G^{5}}{(1-\beta-\beta G)\left[(\beta-1)^{2}+\beta^{2} G^{2}\right]}<\frac{(1-\beta)^{2}}{\beta^{2} G^{2}+(1-\beta-\beta G)}
$$

or equivalently,

$$
(\beta-1)^{4} \beta^{2} G^{2}-\beta^{6} G^{7}-(1-\beta-\beta G) \beta^{4} G^{5}<(1-\beta-\beta G)(1-\beta)^{2} \beta^{2} G^{2}
$$

This inequality can be further simplified to $\beta G^{3}\left(\beta^{2} G^{2}+1-\beta-\beta G\right)>(1-\beta)^{2}(G+\beta-1)$, which has been established above.

The third iterate of $f(\cdot)$ is given by

$$
f^{3}(x)= \begin{cases}G^{1+\alpha+\alpha^{2}}(x)^{\alpha^{3}} & \text { if } 0 \leq x \leq x_{21} \equiv G^{-(\alpha+1) / \alpha^{2}}, \\ \frac{\beta G^{2+\alpha} x^{\alpha^{2}}}{\beta-1+G^{\alpha+1} x^{\alpha^{2}}} & \text { if } x_{21} \leq x \leq x_{11} \\ \frac{\beta^{2} G^{3} x^{\alpha}}{(\beta-1)^{2}+(\beta-1+\beta G) G x^{\alpha}} & \text { if } x_{11} \leq x \leq x_{22} \equiv\left(\frac{1-\beta}{G-\beta G^{2}}\right)^{1 / \alpha} \\ G\left(\frac{\beta G^{2} x^{\alpha}}{\beta-1+G x^{\alpha}}\right)^{\alpha} & \text { if } x_{22} \leq x \leq 1 \\ G\left(\frac{\beta^{2} G^{2} x}{(\beta-1)^{2}+(\beta-1+\beta G) x}\right)^{\alpha} & \text { if } 1 \leq x \leq x_{23} \equiv \frac{(1-\beta)^{2}}{\beta^{2} G^{2}+(1-\beta-\beta G)} \\ \frac{\beta^{3} G^{3} x}{(\beta-1)^{3}+\left[(\beta-1)^{2}+(\beta-1) \beta G+\beta^{2} G^{2}\right] x} & \text { if } x_{23} \leq x \leq x_{12} \\ \frac{\beta^{1+\alpha} G^{2+\alpha} x^{\alpha}}{(\beta-1)(\beta-1+x)^{\alpha}+\beta^{\alpha} G^{1+\alpha} x^{\alpha}} & \text { if } x_{12} \leq x \leq G\end{cases}
$$

Next, the fourth $f^{4}(\cdot)$ iterate of $f(\cdot)$ is given by

$$
f^{4}(x)= \begin{cases}G^{1+\alpha+\alpha^{2}+\alpha^{3}}(x)^{\alpha^{4}} & \text { if } 0 \leq x \leq x_{31} \equiv G^{-\left(\alpha^{2}+\alpha+1\right) / \alpha^{3}}, \\ \frac{\beta G^{\alpha^{2}+\alpha+2} x^{3}}{\beta-1+G^{\alpha^{2}+\alpha+1} x^{\alpha^{3}}} & \text { if } x_{31} \leq x \leq x_{21} \\ \frac{\beta^{2} G^{\alpha+3} x^{\alpha^{2}}}{(\beta-1)^{2}+(\beta-1+\beta G) G^{\alpha+1} x^{\alpha^{2}}} & \text { if } x_{21} \leq x \leq x_{32} \equiv\left(\frac{1-\beta}{(1-\beta G) G^{\alpha+1}}\right)^{1 / \alpha^{2}} \\ G\left(\frac{\beta G^{\alpha+2} x^{\alpha^{2}}}{\beta-1+G^{\alpha+1} x^{\alpha^{2}}}\right)^{\alpha} & \text { if } x_{32} \leq x \leq x_{11} \\ G\left(\frac{\beta^{2} G^{3} x^{\alpha}}{(\beta-1)^{2}+(\beta-1+\beta G) G x^{\alpha}}\right)^{\alpha} & \text { if } x_{11} \leq x \leq x_{33} \equiv\left(\frac{(1-\beta)^{2}}{\beta^{2} G^{3}+(1-\beta-\beta G) G}\right)^{1 / \alpha} \\ \frac{\beta^{3} G^{4} x^{\alpha}}{(\beta-1)^{3}+\left[(\beta-1)^{2}+(\beta-1) \beta G+\beta^{2} G^{2}\right] G x^{\alpha}} \\ \frac{\beta^{1+\alpha} G^{2+2 \alpha} x^{\alpha^{2}}}{(\beta-1)\left(\beta-1+G x^{\alpha}\right)^{\alpha}+\beta^{\alpha} G^{1+2 \alpha} x^{\alpha^{2}}} & \text { if } x_{33} \leq x \leq x_{22} \leq x \leq 1 \\ \frac{\beta^{1+2 \alpha} G^{2+2 \alpha} x^{\alpha}}{(\beta-1)\left[(\beta-1)^{2}+(\beta G+\beta-1) x\right]^{\alpha}+\beta^{2 \alpha} G^{1+2 \alpha} x^{\alpha}} & \text { if } 1 \leq x \leq x_{23} \\ \frac{\beta^{4} G^{4} x}{(\beta-1)^{4}+\left[(\beta-1)^{3}+\beta G(\beta-1)^{2}+\beta^{2} G^{2}(\beta-1)+\beta^{3} G^{3}\right] x} & \text { if } x_{23} \leq x \leq x_{34} \equiv \frac{(1-\beta)^{2}-\beta^{3} G^{3}-(1-\beta-\beta G) \beta G}{\left(\frac{\beta^{3} G^{3} x}{\left.(\beta-1) \beta G+\beta^{2} G^{2}\right] x}\right)^{\alpha}} \\ G\left(\frac{\text { if } x_{34} \leq x \leq x_{12}}{(\beta-1)^{3}+\left[(\beta-1)^{2}+(\beta-1)\right.}\right. \\ G\left(\frac{\beta^{1+\alpha} G^{2+\alpha} x^{\alpha}}{\left.(\beta-1)(\beta-1+x)^{\alpha}+\beta^{\alpha} G^{1+\alpha x^{\alpha}}\right)^{\alpha}}\right. & \text { if } x_{12} \leq x \leq \min \left\{G, x_{35} \equiv \frac{(1-\beta) x_{22}}{x_{22}-\beta G}\right\} \\ \frac{\beta^{2+\alpha} G^{3+\alpha} x^{\alpha}}{(\beta-1)^{2}(\beta-1+x)^{+}+(\beta-1+\beta G) \beta^{\alpha} G^{1+\alpha} x^{\alpha}} & \text { if } x_{35} \leq x \leq G \text { and } x_{35}<G\end{cases}
$$

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Figure 1: The Second Iterate of the M-map for $\tau<1$


Figure 2: The Third Iterate of the M-map for $\tau<1$


Figure 3: The Fifth and Sixth Iterates of the M-map for $\tau=1$


Figure 4: A Counter-example


Figure 5: The Fourth Iterate of the M-map for $\tau<1$ and $\rho<1$


Figure 6: Example 1 on the Existence of a 7-Period Cycle


Figure 7: KP-construction for the M-map


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[^1]:    ${ }^{1}$ See [1, p. 91]. The original is titled "The butterfly effect" and constitutes Appendix One in the author's The Essence of Chaos. Seattle: University of Washington Press.
    ${ }^{2}$ See [1, p. 8]. The original is from "Finding a horseshoe on the beaches of Rio," The Mathematical Intelligencer 20, 39-44.
    ${ }^{3}$ See [1 p. 89]. In a section titled "The Chaos Revolution: 1968-1998", the author gives a list of 23 paradigm shifters that includes only one economist: Richard Goodwin.
    ${ }^{4}$ Ruette continues, "It relies generally on the idea of unpredictability or instability, i.e. knowing the trajectory is not enough to know what happens elsewhere." The sentiment expressed in these sentences is part of the folklore of the subject; we cull them from [51, p. iv].
    ${ }^{5}$ The adjectives "dynamic and "stagnant" bow to the conventional categorization: in the Solow regime, the economy is growing through capital accumulation at an exogenous rate of population growth, and in the Romer regime, through the expansion of product varieties at an endogenously-generated growth rate.
    ${ }^{6}$ See Figure 1. Econometricians will note and appreciate the distinction between the standard logistic function and the formula for the right arm given in Equation 1 below. The former is given by $1 /(a+$

[^2]:    ${ }^{9}$ To be sure, this is our subjective, and perhaps opinionated reading of 42; only an individual author knows his or her motivation.
    ${ }^{10}$ The uninitiated reader should note that all these papers involve existence of the relevant cycle, and that their concern is not with stability.
    ${ }^{11}$ For this literature, see [16, [14, 41, [60 and their references.
    ${ }^{12}$ As we observe below, this case has especially caught the attention of the growth and development literature; see [16], 60].
    ${ }^{13}$ It is important, given 44, that the reader not give more than the necessary weight to our claim:

[^3]:    ${ }^{15}$ For an exposition of Stefan's 1977 construction of the square root of a map, see [51, Example 3.22].

[^4]:    ${ }^{16}$ See [39, Footnote 8]. The footnote is an important marker of the professional understanding of dynamical systems at the time. It continues, "Another difficulty is that the Schwartzian derivative of the map, $\Phi$, is not negative, which means, among other things, that the iteration of the critical point, $\Phi^{\prime}\left(k_{c}\right)$, may fail to detect stable cycles, even if they exist." Stability of the cycles is not our (or Li-Yorke's) concern in this essay, at least until Section 7.

[^5]:    ${ }^{17}$ The authors then discuss their Figure 2 in this context. Indeed, numerical proofs abound in the subject. In [45, p. 238], the author proves his Claim 8 through his Figure 1. For numerical justifications, see Footnotes 42 and 44 .
    ${ }^{18}$ Indeed, one of the 3 -period cycles is stable, though stability is not the concern here.

[^6]:    ${ }^{19}$ The authors do not write out the straightforward proof, and we do so only for completeness and for the reader's convenience. We also take this opportunity to thank two anonymous referees for seeing the need to couch our arguments in terms of the absorbing interval.
    ${ }^{20}$ In this, the M-map shares a commonality with the check-map studied in, for example, [28, 29, 30]. We shall have a little more to say regarding the latter in the concluding remarks.

[^7]:    ${ }^{21}$ In this comparison, note the fact that there is no continuum of 4-period cycles as analytically revealed by the algebraic specification of the fourth iterate. We exaggerate this comparison by giving more curvature to the representation of the iterate in the interval ranges $\left[x_{22}, 1\right]$ and $\left[x_{12}, x_{35}\right]$. Also compare Figure 6 with the stylized Figure 7d below.

[^8]:    ${ }^{22}$ See, for example, [30, p. 413] for a discussion of the theorem in the context of the RSS model and the checkmap.
    ${ }^{23}$ The Robinson-Solow-Srinivasan model regarding which there has accumulated a considerable literature; see for example [29] and their references.
    ${ }^{24}$ These pull-back trajectories, and a construction based on them, is surely well-understood and wellknown to workers in dynamical-systems - we use the abbreviation here not to give any mathematical priority but simply as a mnemonic device that has seen an economic application.
    ${ }^{25}$ Indeed, as established in [45], it converges to its unique 2-period cycle for some parameter values, and does not converge for others.

[^9]:    ${ }^{26}$ The answers that follow are heavily indebted to Ruette's 2017 exposition.
    ${ }^{27}$ This constitutes [51, Definition 3.19], and we refer to her Section 3.3 for additional detail and discussion; also 38 and 22].
    ${ }^{28}$ In 42, Footnote 4], he gives "reason why one should not define topological chaos in terms of an uncountable scrambled set."
    ${ }^{29}$ See, in particular, Propositions 2.2 and 2.3, and Footnotes 5 and 6, in 42. Theorem 5 below can be read as a more up-to-date elaboration of his work.

[^10]:    ${ }^{30}$ For recent work that interprets the size of the scrambled set in terms of Lebesgue measure, rather than topological or set-theoretic notions, see [31, 32], but note the typographical error in [32, Section 2, line 3]. Also see 51, Section 6.1].
    ${ }^{31}$ References [55, 58] on how transitivity already implies density and sensitivity for interval maps, have by now a canonical status. For a comprehensive treatment of transitivity, the interested reader is referred to [33]; also [2, 24]. The uninitiated reader will find the tutorial [38] to be a useful starting point.

[^11]:    ${ }^{32}$ In two paragraphs preceding her presentation of the theorem, Ruette charts its evolution from Sharkovsky through Block, Bowen-Franks, and Misurewicz-Szlenk, to Misurewicz in 1978.
    ${ }^{33}$ We refer the economist-reader to the Majumdar-Mitra tutorial [38] as one possible starting point; its emphases remains up-to-date and surprisingly modern. For topological entropy, see [51, Chapter 4] and her references.
    ${ }^{34}$ The first assertion is due to a Kuchta-Smítal in 1987, and the second to Junková-Smítal in 1986; see Theorems 5.33 and 5.17 respectively in 51.
    ${ }^{35}$ The forward orbit of an eventually homoclinic point only asymptotically converges to a periodic point. A good intuitive discussion of homclinicity for smooth maps is available in [11] and in [22], and for piecewise continuous maps, in [51, Section 4.3]. Block saw the relevance of the idea in a 1978 paper.
    ${ }^{36}$ The example that Li-Yorke chaos is strictly weaker has only been shown in 2005 by Ruette; see Section 7.3.2 and Theorem 7.13 in 51.
    ${ }^{37}$ The characterization is a 1991 result of Franzová-Smítal; see Theorem 5.39 for the result and Section 5.6 for the definition of topological sequence entropy, both in [51]. Also see [17] for recent work.
    ${ }^{38}$ See Theorem 7.7 and Theorem 7.8, both in [51] ; also [42] for the importance of the second iterate.

[^12]:    ${ }^{39}$ We have not mentioned, for example, generic and dense chaos [51, Section 6.1] or mean chaos [25, 18, 21 .
    ${ }^{40}$ For references to the relevant papers of Lasota-Yorke, Li-Yorke and Kowalski, see 61. The quotations are taken from the introduction and conclusion of their papers.

[^13]:    ${ }^{41}$ See the second full paragraph on page 542 in 19 for the quote, Figure 2 for the bifurcation diagram, Figures 3 for numerical snapshots of the $2^{n d}$-iterate, and Figures 4 and 5 for the $4^{t h}$-iterate.
    ${ }^{42}$ This is why our own numerical example at the end of Section 3.2 can hardly be considered decisive for the point it attempts to make. This is by now well-understood in the literature on numerical analysis; see 37 for references and further discussion.
    ${ }^{43}$ This equation is identified in 45] and represented in the sentence following (4) in [19]. We remind the reader that $\rho(1)>1$ is a necessary and sufficient condition for the $12^{t h}$ arm of the $4^{\text {th }}$-iterate.
    ${ }^{44}$ See Lozi 37; he concludes, "We have shown ... that it is very difficult to trust in numerical solution of chaotic dynamical dissipative systems. In some cases one can even proof (sic) that it is never possible to obtain reliable results."
    ${ }^{45}$ In this context, we can refer to the open question as to whether the result in 32 can be obtained as a corollary to Mané's 1985 theorem for smooth maps.
    ${ }^{46}$ Specifically, the interesting conjugacy claim and that of positive topological entropy in [19, Proposition 2] is adduced by results in [11] and [20 that pertain to smooth maps rather than to those with a kink, the signature of the M-map. This issue is reconsidered in [57], but the results now appealed to are those of Marotto which again require differentiability of the map; see [57] for reference to the 1978 paper of Marotto and its 2005 correction. [20] is justifiably not cited in [57] which limits attention to non-smooth maps.
    ${ }^{47}$ As is well-known, this was pioneered by Nusse-Yorke 47]; see [23] and [48] for earlier application and subsequent elaboration.

[^14]:    ${ }^{48}$ The panels of Figure 7 show how far one can get with the KP technique developed for the RSS model [28] in [30]. This point is even more evident in maps forged by the addition of a third arm to the M-map, pursued in 41] or those which embed the Matsuyama production structure in an uncertain environment, as in [14.

