



On Mitra's sufficient condition for topological chaos: Seventeen years later[☆]



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HIGHLIGHTS

- An easy and useful extension of Mitra's 2001 theorem on Li–Yorke chaos.
- An alternative proof of Mitra's 2001 result on the Matsuyama model.
- An application of the extended theorem to generalize a result on the RSS model.
- An application of the LMPY theorem to generalize a result on the Matsuyama model.
- Open questions and methodological concerns.

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ABSTRACT

This letter reports an easy extension of Mitra's "easily verifiable" sufficient condition for topological chaos in unimodal maps, and offers its application to reduced-form representations of two economic models that have figured prominently in the recent literature in economic dynamics: the check- and the M-map pertaining to the 2-sector Robinson–Solow–Srinivasan (RSS) and Matsuyama models respectively. A consideration of the iterates of these maps establishes the complementarity of the useful 2001 condition with the 1982 (LMPY) theorem of Li–Misiurewicz–Pianigiani–Yorke when supplemented by a geometric construction elaborated in Khan–Piazza (2011).

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In either version of kaleidics, Keynesian or Viennese, the analytical stress is placed on treating time seriously and not just notionally, which leads in turn to recognition that economic processes are better treated as turbulent than as equilibrated.¹

Richard E. Wagner (2011)

One establishes the failure of Newtonian determinism by using Newton's own equations. The coin-flipping syndrome is pervasive. "Sensitive dependence on initial conditions" has become a catchword of modern science.²

Smale (1998)

1. Introduction

Mitra's (2001) pioneering contribution to topological chaotic dynamics of more than a decade and a half ago is widely cited though perhaps not as widely read. In this letter, we provide a reading, both retrospective and prospective, that extends his easily

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¹ See his "Viennese Kaleidics: Why Liberty and not Policy Calms Turbulence," Presidential address, Society for the Development of Austrian Economics, Washington, DC, 19–21 November 2011.

² See "Finding a horseshoe on the beaches of Rio," *The Mathematical Intelligencer* 20, 39–44.

verifiable sufficient condition, and uses it to generalize results available in Khan and Mitra (2005, 2013) and Deng and Khan (2018) on the Robinson–Solow–Srinivasan (RSS) and Matsuyama (1999) growth models respectively. Our reading, in highlighting a 1982 result³ of Li–Misiurewicz–Pianigiani–Yorke (LMPY), brings out the complementarity of the two results: both have their place in the tool-kit of economic dynamics. The LMPY theorem offers a sufficient condition for odd-period cycles but like the 1975 Li–Yorke theorem, one phrased in terms of existential qualifiers, whereas Mitra's result requires a simple comparison of the third iterate of the turning point of a continuous unimodal map to its fixed point, and is thereby more easily verifiable.

Three additional observations stem from our reading of the 2001 paper. First, it shifts attention from *topological chaos* to *topological entropy*, and thereby to the 1971 Ruelle–Takens notion of *turbulence*, and argues “why one should not define topological chaos in terms of an uncountable scrambled set.”⁴ Second, this shift of emphasis flows naturally into two other concepts: Block's (1978) notion of *homoclinicity*, and Misiurewicz' (1978) modification of Smale's *horseshoe* for two-dimensional smooth homeomorphisms.⁵ Third, the paper's silence in its introduction to a seminal 1999 misreading of the Li–Yorke theorem is a testimony to how far ahead of its time it was in its understanding of the literature on dynamical systems.⁶

2. Two canonical maps

We present reduced-form representations of two canonical models, beginning with the Robinson–Solow–Srinivasan (RSS)-model, studied in Khan and Mitra (2005, 2013), Mitra et al. (2006), Khan and Piazza (2011) and their references. The dynamics of such a model are investigated through the check-map, as described by

$$\phi_c(x) = \begin{cases} f_c(x) = 1 - \xi x & 0 \leq x \leq 1/\xi \\ g_c(x) = (1-d)x - (1-d)/\xi & 1/\xi \leq x \leq 1, \end{cases}$$

where $\xi > 1$ and $0 < d < 1$. See Fig. 1a, Khan and Mitra (2013) and their references for a geometric representation. Note that for parameter values out of this range, the dynamics are tame and easily understood.⁷

Next, we turn to the Matsuyama (M)-model, studied in Matsuyama (1999), Mitra (2001), Mukherji (2005), Gardini et al. (2008), Deng and Khan (2018) and their references. The dynamics of such a model are investigated through the M-map as described by

$$\phi_m(x) = \begin{cases} f_m(x) = Gx^\alpha & 0 \leq x \leq 1 \\ g_m(x) = \frac{G\beta x}{\beta - 1 + x} & x > 1, \end{cases}$$

³ This paper is the last of a series of direct generalizations of the Li–Yorke theorem by the number theorists, Nathanson and Fuglister; see Deng and Khan (2018) for discussion and references.

⁴ See the reference to the Li–Yorke criterion, rather than to Li–Yorke chaos; Footnote 4, Mitra (2001).

⁵ Later work would only state “This condition ... is exactly the condition for which the fixed point has homoclinic trajectories”, and later on, “the sufficient condition stated by Mitra ... can be enforced in terms of homoclinic trajectories”; see pp. 542, 549, Gardini et al. (2008). For *homoclinicity*, see Grandmont (2008) for smooth maps, and Section 4.3, Ruette (2017) for piecewise continuous maps.

⁶ See Footnote 11, Deng and Khan (2018), for details and discussion of Footnote 8, Matsuyama (1999).

⁷ Whereas our emphasis is on the two-sector RSS model, note that the check-map can also be given totally different conceptual underpinnings, as in Denekere and Judd (1992), a neglected reference.

where $\alpha \in (0, 1)$, $\beta = \alpha^{\frac{\alpha}{1-\alpha}}$ with β in $(1/e, 1)$ and decreasing with α , and $1 < G < (1/\beta) - 1$. See Fig. 1b and Matsuyama (1999) for a diagrammatic representation, and Deng and Khan (2018) for the notation adopted here. We shall confine ourselves to the parameter values

$$\phi_m(G) = \phi_m^2(1) = \frac{\beta G^2}{(\beta - 1 + G)} \equiv \tau < 1.$$

For values outside this range, the dynamics have been well-understood since Matsuyama's pioneering analysis; a cyclical growth pattern occurs when $G \in (1, (1/\beta) - 1)$ and $\beta < 1/2$ (equivalently, $\alpha > 1/2$).

3. Two theorems: LMPY (1982) and Mitra (2001)

The main theorem of Li et al. (1982) on odd-period cycles, as modified in Ruette (2017), can be stated as follows.

Theorem 1. Let f be a continuous map from an interval $[a, b]$ to itself, $0 \leq a < b < \infty$. Let $p \geq 3$ be an odd integer. If for some $x \in [a, b]$, we have either $f^p(x) \leq x < f(x)$ or $f^p(x) \geq x > f(x)$, then f admits a p -period cycle.

Next, we turn to an extension of Mitra's sufficient condition, Proposition 2.3, in Mitra (2001). We need the following notation. Let \mathcal{H} to be the set of continuous maps from an interval, $[a, b]$, to itself, with a generic element h , satisfying the following two conditions

1. There is m in (a, b) , with the map h strictly increasing on $[a, m]$ and strictly decreasing on $[m, b]$.
2. $h(a) \geq a$, $h(b) < b$, and $h(x) > x$ for all x in $(a, m]$.

Let \mathcal{G} to be the set of continuous maps from an interval, $[a, b]$, to itself such that its generic element g satisfies the first condition above by the substitution of “increasing” for “decreasing” and “decreasing” for “increasing”, and the second condition as $g(a) > a$, $g(b) \leq b$, and $g(x) < x$ for all x in $[m, b]$.

We shall also need the following definition; also see Chapter 2, Block and Coppel (1992). A map f is said to be turbulent if there exist three points, c_1 , c_2 , and c_3 , in $[a, b]$ such that $f(c_3) = f(c_1) = c_1$ and $f(c_2) = c_3$ with either $c_1 < c_2 < c_3$ or $c_3 < c_2 < c_1$.

We now provide an extension of Mitra's theorem; see Section 7 below for a proof.

Theorem 2. Let $f \in \mathcal{G} \cup \mathcal{H}$ and let z be the unique interior fixed point of f . f^2 is turbulent and f exhibits topological chaos, if the following condition is satisfied: $f \in \mathcal{H}$ with $f^2(m) < m$ and $f^3(m) \leq z$; or $f \in \mathcal{G}$ with $f^2(m) > m$ and $f^3(m) \geq z$.

4. The check-map and topological chaos

Khan and Mitra (2005, 2013) prove that the check-map exhibits topological chaos in what they term to be the *borderline* case of the 2-sector RSS model: for parameter values given by $(\xi - 1/\xi)(1 - d) = 1$. In light of our extension of Mitra's sufficient condition, Theorem 2 above, we can go substantially beyond the knife-edge parameter values determined earlier. Following the terminology in Khan and Mitra (2013), the corollary below establishes that the check-map exhibits topological chaos for both *borderline* and the *outside* cases.

Corollary 1. The check-map exhibits topological chaos if $(\xi - 1/\xi)(1 - d) \geq 1$.

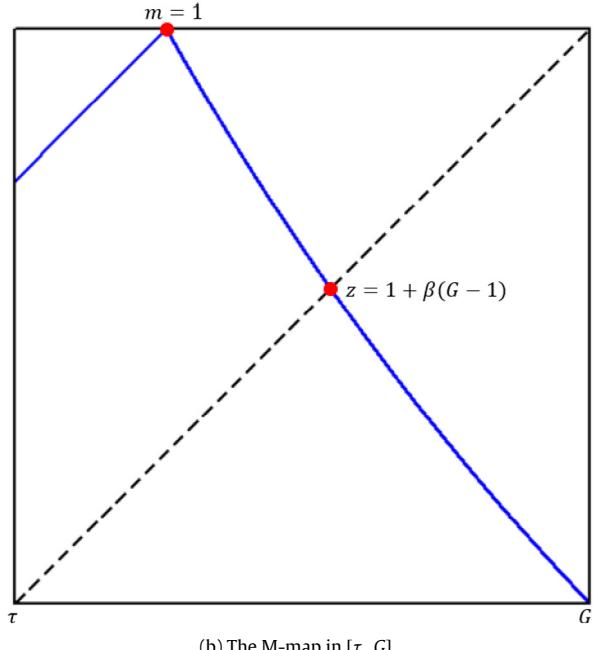
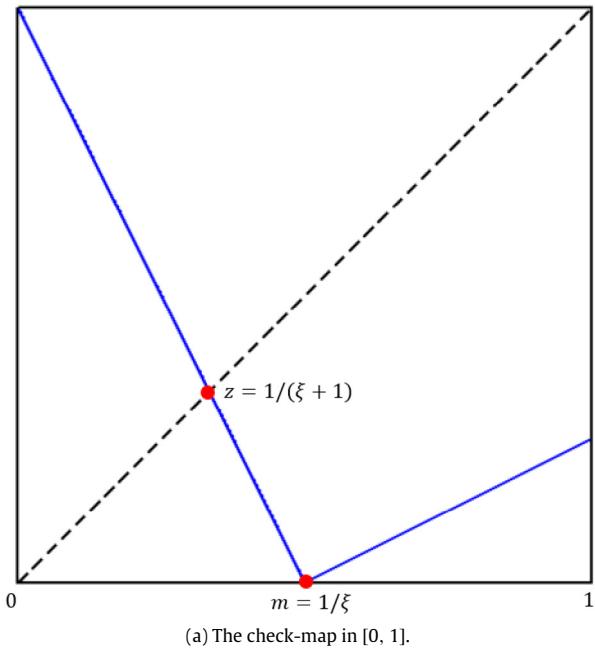


Fig. 1. Two canonical maps in their trapping squares.

Proof. The check-map belongs to \mathcal{G} with $m = 1/\xi$. Since $\xi > 1$, $\phi_c^2(1/\xi) = 1 > 1/\xi$. Since $(\xi - 1/\xi)(1 - d) \geq 1$, we must have $\phi_c^3(1/\xi) = (1-d)(1-1/\xi) \geq 1/(1+\xi)$, where $1/(1+\xi)$ is the fixed point of the check-map. According to [Theorem 2](#), the check-map then exhibits topological chaos. ■

Remark 1. Note that [Mitra's 2001](#) condition cannot be used to prove the result in [Khan and Mitra \(2005\)](#). However, their proof does not require knowledge of the fact that the optimal policy function is given by the check-map, and therefore of interest from that count alone.

Remark 2. An alternative way to establish topological chaos for the outside case, $(\xi - 1/\xi)(1 - d) > 1 \iff \phi_c^3(1/\xi) = (1-d)(1-1/\xi) >$

$1/(1 + \xi)$, is to apply [Theorem 1](#) directly. We offer a proof in [Section 7](#) below. Whereas clearly of interest in the development of ideas, this application does not allow us to cover the *borderline* case.

5. The M-map and topological chaos

[Mitra \(2001\)](#) shows that the M-map exhibits topological chaos if his Condition 2.13, $G\tau^\alpha < 1 + \beta(G - 1) \equiv z$ is satisfied, z being the fixed point of the M-map. It is clear that [Theorem 2](#) allows a direct generalization of this result by also admitting the knife-edge equality. However, in this section, we provide a direct proof of the original result as a consequence of [Theorem 1](#). Such a proof utilizes the iterates, supplemented by the KP-construction, presented in [Deng and Khan \(2018\)](#). It also enables a strict generalization of their [Theorem 3](#).

Towards these ends, we introduce the following notation for $n \geq 3$.

$$\begin{aligned} \varphi_n(\beta, G) &\equiv g^{n-3}(f(g^2(1))) \\ &= \left[\left(\frac{\beta G}{\beta - 1} \right)^{n-3} G\tau^\alpha \right] \left[1 + \frac{\left(\frac{\beta G}{\beta - 1} \right)^{n-3} - 1}{1 + \beta G - \beta} G\tau^\alpha \right]^{-1} \end{aligned}$$

$$\mathcal{P}_n \equiv \{(\beta, G) \in (1/e, 1/2) \times (1, (1-\beta)/\beta) : \varphi_n(\beta, G) \in (0, 1)\}.$$

The following results are of general interest for the M-map; see [Section 7](#) below for proofs.

Lemma 1. (i) $\mathcal{P}_3 = \mathcal{P}_5 = \emptyset$; (ii) $\mathcal{P}_{2n-1} \subseteq \mathcal{P}_{2n+1}$ for any $n > 1$.

[Theorem 3](#) in [Deng and Khan \(2018\)](#) asserts the existence of a 7-period cycle if $(\beta, G) \in \mathcal{P}_7$, and that $\mathcal{P}_7 \neq \emptyset$. The following is a natural generalization.

Theorem 3. For any $n > 1$, there exists a $(2n + 1)$ -period cycle if $(\beta, G) \in \mathcal{P}_{2n+1}$.

Now the result in [Mitra \(2001\)](#) follows as a direct and easy corollary.

Corollary 2. The M-map exhibits topological chaos if $G\tau^\alpha < 1 + \beta(G - 1)$.

6. Several open questions

The reading of [Mitra \(2001\)](#) provided above naturally leads one to ask for complete characterizations of the phenomena that it reads: it asks for necessary and sufficient conditions for odd- and even-period cycles in continuous unimodal and bimodal maps, and more particularly, for conditions that involve specific magnitudes such as 2^∞ . This is part of ongoing work to be reported in the near future.

We conclude this letter with a consideration of two views of a rather obvious epistemological significance that are increasingly pervasive in the literature: (i) mathematical results pertaining to smooth non-linear maps carry over, more or less, to smooth, leave alone linear, maps with kinks, (ii) the success of experimental mathematics and of high-speed computing renders “numerical proofs” substitutable for (formal and traditional) mathematical proofs. Some further elaboration of our strong contestation of these views may be worthwhile for future research.

The field of chaotic dynamics was initiated by the so called “butterfly effect” whose very point was that an infinitesimal change in initial conditions can lead to large appreciable changes over time;

and as such, approximation and numerical computation constitute a double-edged sword. While clearly of importance for generating fruitful conjectures, such procedures need buttressing by fully rigorous proofs.

For concreteness, consider, for example, the criticism by Gardini et al. (2008) of Mukherji's (2005) claim of stability of 4-period cycles.⁸ Since this is established through a bifurcation diagram based on numerical procedures not made explicit, one may legitimately continue to regard this dissonance between the two papers as an unresolved open question. More specifically, the interesting claims of conjugacy and positive topological entropy in Proposition 2 in the same paper is adduced by invoking results in Devaney, Gardini and their followers, but these pertain to a smooth map rather than those with a kink, the very signature of the M-map. As such, these claims must also be regarded as open.⁹

7. Proofs of the results

Proof of Theorem 2. For $f \in \mathcal{H}$, Mitra (2001) has shown that if $f^2(m) < m$ and $f^3(m) < z$, then f^2 is turbulent and f exhibits topological chaos. We focus on $f^3(m) = z$, and prove that f^2 is turbulent. By definition of \mathcal{H} , we know $m < z$. Given $f^2(m) < m$, we obtain $f^2(m) < m < z$. Since $f^3(m) = z$, $f^2(f^2(m)) = f(z) = z = f^2(z)$. Since $f^2(m) = f^2(z)$ trivially, f^2 is turbulent and f exhibits topological chaos.

We now turn to the hypothesis that $f \in \mathcal{G}$ with $f^2(m) > m$ and $f^3(m) \geq z$. Consider a map $\tilde{f} : [a, b] \rightarrow [a, b]$, defined as $\tilde{f}(x) = a + b - x$ for x in $[a, b]$. Let $h \equiv \tilde{f}^{-1}$ of \tilde{f} . Since \tilde{f} is one-to-one, onto, and continuous, and \tilde{f}^{-1} is continuous, \tilde{f} is homeomorphism. According to Ruette (2017), f and h are topologically conjugate. Since f is in \mathcal{G} , there exists m in (a, b) such that f strictly decreases on $[a, m]$ and strictly increases on $[m, b]$. By construction, we must have h strictly increases on $[a, a + b - m]$ and strictly decreases on $[a + b - m, b]$. Since f is in \mathcal{G} , $f(a) > a$, $f(b) \leq b$, and $f(x) < x$ for all x in $[m, b]$. Again, by construction, this implies $h(a) \geq a$, $h(b) < b$, and $h(x) > x$ for all x in $(a, a + b - m]$. Therefore, h is in \mathcal{H} . Since $f^2(m) > m$, $h^2(a + b - m) = a + b - f(a + b - h(a + b - m)) = a + b - f^2(m) < a + b - m$. Since $f^3(m) \geq z$, $h^3(a + b - m) = a + b - f^3(m) \leq a + b - z$, where $a + b - z$ is the fixed point of h . Then applying what we have shown above for \mathcal{H} , h^2 is turbulent and h exhibits topological chaos. Since f and h are topologically conjugate, f^2 is also turbulent and f exhibits topological chaos. ■

Proof of the Claim in Remark 1. First, since $\phi_c^3(1/\xi) = (1-d)(1-1/\xi)$, if $(1-d)(1-1/\xi) \geq 1/\xi$, we have $\phi_c^3(1/\xi) \geq 1/\xi > 0 = \phi_c(1/\xi)$. According to Theorem 1, there exists a 3-period cycle and therefore the check-map exhibits topological chaos. Now consider $\phi_c^3(1/\xi) \in (1/(1+\xi), 1/\xi)$. We claim that there exists a $(2k+3)$ -period cycle for $\phi_c^3(1/\xi)$ in $\left[\frac{1+(1/\xi)^{2k+1}}{1+\xi}, \frac{1+(1/\xi)^{2k-1}}{1+\xi}\right]$ with k being a natural number. To see this, define

$$z_n \equiv f_c^{n-3}(\phi_c^3(1/\xi)) = \frac{1 - (-\xi)^{n-3}}{1 + \xi} + (-\xi)^{n-3}\phi_c^3(1/\xi)$$

for $n > 3$. If $\phi_c^3(1/\xi)$ is in $\left[\frac{1+(1/\xi)^{2k+1}}{1+\xi}, \frac{1+(1/\xi)^{2k-1}}{1+\xi}\right]$, then we have $z_4 < 1/\xi$, $z_5 < 1/\xi$, ..., $z_{2k+1} < 1/\xi$, $z_{2k+2} < 1/\xi$, and $z_{2k+3} \geq 1/\xi$, among which the first $(2k-2)$ inequalities guarantee that $z_{2k+3} = \phi_c^{2k+3}(1/\xi)$. Therefore, $\phi_c^{2k+3}(1/\xi) \geq 1/\xi > f(1/\xi)$.

⁸ The authors assert (pp. 542, 548) that "transition to chaos may occur via the standard period-doubling bifurcation sequence, writing about a stable 4-cycle found after the stabilization of the 4-cycle, which we show not possible (sic). It turns out that a stable 4-cycle is impossible".

⁹ See the trail of proofs of the Proposition 2 in Gardini et al. (2008) that ends in Marotto's useful 2005 correction.

Again, according to Theorem 1, there exists a $(2k+3)$ -period cycle. Since we have

$$\bigcup_{k=1}^{\infty} \left[\frac{1 + (1/\xi)^{2k+1}}{1 + \xi}, \frac{1 + (1/\xi)^{2k-1}}{1 + \xi} \right) = (1/(1 + \xi), 1/\xi),$$

if $\phi_c^3(1/\xi)$ is in $(1/(1 + \xi), 1/\xi)$, then $\phi_c^3(1/\xi)$ must be in $\left[\frac{1+(1/\xi)^{2k+1}}{1+\xi}, \frac{1+(1/\xi)^{2k-1}}{1+\xi}\right)$ for some k , which gives rise to an odd-period cycle and therefore topological chaos for the check-map. ■

Proof of Lemma 1. For (i), $\mathcal{P}_3 = \emptyset$ directly follows Lemma 1 in Deng and Khan (2018) (also see Matsuyama, 1999), and $\mathcal{P}_5 = \emptyset$ directly follows Lemma 2(i).

For (ii), pick (β, G) from \mathcal{P}_{2n-1} . Following the definition of \mathcal{P}_{2n-1} , we have

$$0 < \varphi_{2n-1}(\beta, G)$$

$$\equiv \left[\left(\frac{\beta G}{\beta - 1} \right)^{2n-4} G \tau^\alpha \right] \left[1 + \frac{\left(\frac{\beta G}{\beta - 1} \right)^{2n-4} - 1}{1 + \beta G - \beta} G \tau^\alpha \right]^{-1} \\ \leq 1$$
(1)

Since $0 < \varphi_{2n-1}(\beta, G)$, we must have $1 + \left[\left(\frac{\beta G}{\beta - 1} \right)^{2n-4} - 1 \right] G \tau^\alpha / (1 + \beta G - \beta) > 0$. Rearranging Inequality (1), we obtain

$$0 < \left[\frac{\beta G - \beta}{1 + \beta G - \beta} \left(\frac{\beta G}{\beta - 1} \right)^{2n-4} + \frac{1}{1 + \beta G - \beta} \right] G \tau^\alpha \leq 1.$$

Since $\beta G < 1 - \beta$, we obtain

$$0 < \left[\frac{\beta G - \beta}{1 + \beta G - \beta} \left(\frac{\beta G}{\beta - 1} \right)^{2n-2} + \frac{1}{1 + \beta G - \beta} \right] G \tau^\alpha \leq 1,$$

which can be further simplified to $0 < \varphi_{2n+1}(\beta, G) \leq 1$, or equivalently to $(\beta, G) \in \mathcal{P}_{2n+1}$. Therefore, $\mathcal{P}_{2n-1} \subseteq \mathcal{P}_{2n+1}$, and the proof is complete. ■

Proof of Theorem 3. For $n = 1, 2$, the statement is vacuously true because $\mathcal{P}_3 = \mathcal{P}_5 = \emptyset$. Theorem 3 in Deng and Khan (2018) is special case when $n = 3$, and they have shown $\mathcal{P}_7 \neq \emptyset$. We now prove this theorem by induction. Suppose it is true for $n = 1, 2, 3, \dots, k-1$. We now want to show that it is also true for $n = k$.

Since we know $\mathcal{P}_{2k-1} \subseteq \mathcal{P}_{2k+1}$, we can write \mathcal{P}_{2k+1} as a disjoint union of two subsets: $\mathcal{P}_{2k+1} = \mathcal{P}_{2k-1} \cup (\mathcal{P}_{2k+1} - \mathcal{P}_{2k-1})$. If $(\beta, G) \in \mathcal{P}_{2k-1}$, then there exists a $(2k-1)$ -period cycle and according to Sharkovskii's theorem, there exists a $(2k+1)$ -period cycle.

Consider $(\beta, G) \in \mathcal{P}_{2k+1} - \mathcal{P}_{2k-1}$. We can write $\mathcal{P}_{2k+1} - \mathcal{P}_{2k-1}$ more explicitly as $\{(\beta, G) \in (1/e, 1/2) \times (1, (1-\beta)/\beta) : \varphi_3(\beta, G) > 1, \varphi_5(\beta, G) > 1, \dots, \varphi_{2k-1}(\beta, G) > 1, \varphi_{2k+1}(\beta, G) \in (0, 1]\}$. We now want to show that for any $(\beta, G) \in \mathcal{P}_{2k+1} - \mathcal{P}_{2k-1}$, $\varphi_{2\ell}(\beta, G) > 1$ for $\ell = 2, 3, \dots, k$. Since $\varphi_{2k+1}(\beta, G) \in (0, 1]$, we have

$$0 < \varphi_{2k+1}(\beta, G)$$

$$\equiv \left[\left(\frac{\beta G}{\beta - 1} \right)^{2k-2} G \tau^\alpha \right] \left[1 + \frac{\left(\frac{\beta G}{\beta - 1} \right)^{2k-2} - 1}{1 + \beta G - \beta} G \tau^\alpha \right]^{-1} \\ \leq 1,$$

which implies $G \tau^\alpha < 1 + \beta G - \beta$.

By definition, $\varphi_{2\ell}(\beta, G) = g_m(\varphi_{2\ell-1}(\beta, G))$ for $\ell > 1$. Therefore, $\varphi_{2\ell-1}(\beta, G) > 1$ guarantees $\varphi_{2\ell}(\beta, G) > 0$ for $\ell = 2, 3, \dots, k$.

Suppose for some $\ell, \varphi_{2\ell}(\beta, G) \in (0, 1]$, or equivalently,

$$0 < \left[\left(\frac{\beta G}{\beta - 1} \right)^{2\ell-3} G\tau^\alpha \right] \left[1 + \frac{\left(\frac{\beta G}{\beta - 1} \right)^{2\ell-3} - 1}{1 + \beta G - \beta} G\tau^\alpha \right]^{-1} \leq 1.$$

Notice $\beta G/(\beta - 1) < 0$. Rearranging the inequality, we have

$$\left[\frac{\beta G - \beta}{1 + \beta G - \beta} \left(\frac{\beta G}{\beta - 1} \right)^{2\ell-3} + \frac{1}{1 + \beta G - \beta} \right] G\tau^\alpha \geq 1,$$

which implies $G\tau^\alpha > 1 + \beta G - \beta$, contradicting to what is implied by $\varphi_{2k+1}(\beta, G) \in (0, 1]$.

Therefore, for $(\beta, G) \in \mathcal{P}_{2k+1} - \mathcal{P}_{2k-1}$, we obtain the following set of inequalities: $\varphi_n(G, \beta) > 1$ for $n = 3, 4, \dots, 2k$ and $\varphi_{2k+1} \in (0, 1]$. This set of inequalities suggest that under the original M-map, $\phi_m^n(1) > 1$ for $n = 3, 4, \dots, 2k$, and $\phi_m^{2k+1}(1) \leq 1$. Combined with the fact that $\phi_m(1) = G > 1$, we have $\phi_m^{2k+1}(1) \leq 1 < \phi_m(1)$. According to Theorem 1, this condition guarantees a $(2k + 1)$ -period cycle.¹⁰ Therefore, by induction, we have shown that for any natural number n , a $(2n + 1)$ -period cycle exists if $(\beta, G) \in \mathcal{P}_{2n+1}$. ■

Proof of Corollary 2. If $G\tau^\alpha < 1 + \beta(G - 1)$, then $\varphi_{2n+1}(\beta, G) > 0$ for any natural number n . Define $\epsilon = \frac{1 + \beta(G - 1) - G\tau^\alpha}{2(\beta G - \beta)}$. Since $\beta G < |\beta - 1|$, we can pick a sufficiently large n such that $(\beta G/(\beta - 1))^{2n-2} < \epsilon$, which implies $\varphi_{2n+1}(\beta, G) \leq 1$. According to our theorem, there exists a $(2n + 1)$ -period cycle and therefore the M-map exhibits topological chaos. ■

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¹⁰ The difficulty of this proof arises from the fact that the expression of ϕ_m^{2n+1} can be very complicated as the M-map is only piecewise smooth and nonlinear.