# On growing through cycles: Matsuyama’s M-map and Li-Yorke chaos* 

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## A R T I CLE IN F O

## Article history

Received 24 December 2016
Received in revised form 25 October 2017
Accepted 27 October 2017
Available online 10 November 2017

## Keywords:

Matsuyama's model
Iterations
Li-Yorke chaos
KP-construction
3-period
Cycles


#### Abstract

Recent work of Gardini et al. (2008), building on earlier work of Mitra (2001) and Mukherji (2005), considers the so-called M-map that generates a dynamical system underlying Matsuyama's (1999) endogenous growth model. We offer proofs of the fact that there do not exist 3- or 5-period cycles in the M -map, and an example (a numerical proof) of the existence of a 7-period cycle. We use the latter, and a construction in Khan and Piazza (2011), to identify a range of parameter values of the M-map that guarantee the existence of cycles of all periods, except 3 and 5 . Our argumentation relies on, and reports, the first four iterations of the M-map that may have independent interest.


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The term "chaos" was introduced into mathematics by Li and Yorke in 1975 without formally defining what chaos is. Afterwards, various definitions were proposed. They do not coincide in general and none of them can be considered as the unique "good" definition of chaos. One may ask "What is chaos then?" Ruette (2017)

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## 1. Introduction

In a conceptually-imaginative contribution, Matsuyama (1999) provided a model that generates endogenous growth through the introduction of new varieties chosen from a continuum of commodities, a technological specification pioneered by Rivera-Batiz and Romer (1991); also see the subsequent discussion in Gancia and Zilibotti (2005), Dudek (2010), Yano and Furukawa (2013) and Yano et al. (2013). A particularly attractive feature of the model is its demonstration, under specific parametric restrictions, that an economy may find itself in a stagnant Solow regime, and a dynamic Romer regime, or at alternative, exogenously-specified equal intervals of time, in both. ${ }^{2}$ From a technical point of view, the model can be represented by the, here so-called, M-map: a piecewise smooth two-parameter map, one of whose arms is the (monotonically-increasing) intensive form of the Cobb-Douglas function, and the other, a (monotonically-decreasing) version of the solution to a differential equation involving the logistic map. ${ }^{3}$ The arms are stitched together at a non-differentiable kink,

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Fig. 1. The second iterate of the M-map for $\tau<1$.
subsequently referred to as the critical point of the map. The model is thus of interest for both substantive and technical reasons.

The primary objective of the essay is technical. We present our technical results with the pioneering analysis of Mitra (2001) as the relevant backdrop, and single out the M-map for a deeper analytical treatment. Mitra took Matsuyama's observation that the Mmap cannot have a 3-period cycle as a point of departure to develop a weaker sufficient condition for Li-Yorke chaos, one hinging on the relation of the third iterate of the critical point to its unstable fixed point. ${ }^{4}$ He applied his sufficient condition to establish (also furnish numerical values for) the possibility of the M-map being a Li-Yorke chaotic map. Subsequent work of Mukherji (2005), and of Gardini et al. (2008), followed in Mitra's footsteps not so much as on Li-Yorke chaos, much less on his sufficient condition for it, but on the stability of 2-and 4 -period cycles. They also presented a bifurcation analysis based on a comprehensive numerical investigation. Yano et al. (2011) carried the conversation forward by changing the subject to ergodic chaos.

Generalizations of the Li-Yorke criteria parallel to that of Mitra, have been available in the mathematical literature for quite some time, and the questions this work asks and answers have not been posed for the M-map: in the light of the generalization of the LiYorke theorem in Nathanson (1977), what about the existence or non-existence of 5-and 7-period cycles? And in the light of Fuglister (1979), what about any odd-numbered cycle? And finally, in the light of Li et al. (1982), what about a cycle of order $2^{n}$ for any natural number $n$ ? More to the point, a rigorous mathematical proof of the non-existence of a 3-period cycle, one that would underpin the motivation for Mitra's influential sufficient condition, has still not been offered for a model that is well on its way to becoming an important and canonical marker for the growth and development, as well as the macroeconomic, literature. ${ }^{6}$

[^2]Sections 3 to 5 of this paper address this deficiency. They show that 3- and 5-period cycles do not exist for any feasible parameter values of the M-map, but cycles of all other periods do exist for specific identifiable intervals of parameter values. In particular, they show that one can identify an interval of parameter values for which a 7-period cycle can exist. All this is executed through a presentation of analytical formulae for the first four iterates of the M-map. These iterate specifications serve to delineate the manifold of the 2-dimensional parameter space, and to complement, through additional geometric and algebraic considerations, the numerical analysis in Gardini et al. (2008). In particular, the second iterate is used in Section 2 to give a complete characterization of all iterates for a specific parameter value that allows a 2-period oscillation between the Solow and Romer regimes. ${ }^{7}$ We observe in passing that it is a little surprising that such iterates have not been reported in the literature on economic dynamics: they constitute the secondary objective of the paper. ${ }^{8}$

## 2. The M-map and its second iterate

This tripartite section specifies the M-map arising from Matsuyama's cyclical growth model, and with the devotion of the second iterate of the M-map and to special case, begins the analytical argumentation that constitutes the paper.

### 2.1. The M-map

The basic model is lucidly diagrammed in Matsuyama (1999) (also see Fig. 1 presented here), and the algebra of its dynamics is given by
$x_{t}=f\left(x_{t-1}\right) \equiv \begin{cases}f_{l}\left(x_{t-1}\right)=G x_{t-1}^{1-1 / \sigma} & 0 \leq x_{t-1} \leq 1 \\ f_{r}\left(x_{t-1}\right)=\frac{G x_{t-1}}{1+\theta\left(x_{t-1}-1\right)} & x_{t-1}>1,\end{cases}$
where $\theta \equiv(1-1 / \sigma)^{1-\sigma}, \alpha=1-(1 / \sigma)$, leading to $\sigma>1$ and $\alpha \in(0,1)$. We refer the reader to the economic interpretation of these parameters already available in the literature, ${ }^{9}$ and move on to a transformation of variables whereby the M-map can be rewritten in terms of the pair ( $G, \beta$ ),
$f(x)= \begin{cases}f_{l}(x)=G x^{\alpha} & 0 \leq x \leq 1 \\ f_{r}(x)=\frac{G \beta x}{\beta-1+x} & x>1,\end{cases}$
where $\beta=1 / \theta=\alpha^{\frac{\alpha}{1-\alpha}}$ with $\beta$ in $(1 / e, 1)$ and decreasing with $\alpha$, and $1<G<(1 / \beta)-1$. As brought out in Matsuyama (1999) and Gardini et al. (2008), a cyclical growth pattern occurs when $G \in(1,(1 / \beta)-1)$ and $\beta<1 / 2$ (equivalently, $\alpha>1 / 2$ ). This

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Fig. 2. The third iterate of the M-map for $\tau<1$.
implies that
$f(G)=f^{2}(1)=\frac{\beta G^{2}}{(\beta-1+G)} \equiv \tau<1$.
With $\tau$ as an important benchmark, we turn to the derivation of the second iterate.

### 2.2. The second iterate of the M-map

Consider the M-map given in Fig. 1 (also in Fig. 2) and note how the second arm dips below unity to take the value $\tau$. Two other benchmarks are the elements $x_{11}$ and $x_{12}$ of the set $f^{-1}(1)$, and they are obtained by taking at the vertical determined by the critical point $(1, G)$, and through the use of the 45-degree line, the resulting horizontal to intersect with at the unimodal M-map the two points. Thus, we obtain the four intervals that go towards the determination of the second iterate. We can now algebraically compute it to be

$$
f^{2}(x)= \begin{cases}G^{1+\alpha}(x)^{\alpha^{2}} & \text { if } 0 \leq x \leq x_{11} \equiv G^{-1 / \alpha}  \tag{4}\\ \frac{\beta G^{2} x^{\alpha}}{\beta-1+G x^{\alpha}} & \text { if } x_{11} \leq x \leq 1 \\ \frac{\beta^{2} G^{2} x}{(\beta-1)^{2}+(\beta-1+\beta G) x} & \text { if } 1 \leq x \leq x_{12} \equiv \frac{1-\beta}{1-\beta G} \\ G\left(\frac{\beta G x}{\beta-1+x}\right)^{\alpha} & \text { if } x_{12} \leq x \leq G\end{cases}
$$

Note that $f\left(x_{11}\right)=f\left(x_{12}\right)=1$, leading to $f^{2}\left(x_{11}\right)=f^{2}\left(x_{12}\right)=G$, and that $f^{2}(G)=G \tau^{\alpha}$.

When diagrammed as Fig. 1, we note three intersections of the second iterate with the 45-degree line: the fixed point $\hat{x}$ of the M-map itself, and a 2-period cycle. All this substantiates earlier findings: in particular, we reproduce Theorem 6.1 in Yano et al. (2011) that offers necessary and sufficient conditions for a range of parameter values under which the M-map has the absolute value of its slope greater than unity.

Proposition 1. The M-map, $f:[\tau, G] \rightarrow[\tau, G]$, is expansive, if and only if
$\frac{1}{\alpha}<G<\frac{1-\beta}{2}\left[(\beta+2)+\sqrt{\beta^{2}+4 \beta}\right]$.

We refer the reader to Yano et al. $(2011,2013)$ for the proofs and additional discussion. We now turn to the case $\tau=1$.

### 2.3. A continuum of 2-period cycles

Mukherji (2005) has focused on 2-period cycles, and investigated the special case when $\beta=1 /(1+G)$ as of particular interest. In terms of the notation of this paper, this translates to the parameter value $\tau$ equaling unity, and lead to $x_{12}$ equaling $G$, and the fourth arm of the 2nd-iterate thereby being eliminated. More to the point, we observe its third arm is the identity map. Routine algebra furnishes the following specialization of (1).
$f_{r}(x)=-\frac{G x}{G-(G+1) x}$ and $\frac{d \log f_{r}(x)}{d \log x}=-\frac{f_{r}(x)}{x}$.
This yields the implication that once $x$ enters the absorbing interval $[1, G]$, it alternates between the two points on the right arm of the M-map. Moving on to the specialization of 2nd-iterate, we obtain
$f^{2}(x)= \begin{cases}((1-\beta) / \beta)^{1+\alpha}(x)^{\alpha^{2}} & \text { if } 0 \leq x \leq(\beta /(1-\beta))^{1 / \alpha}, \\ \frac{(1-\beta) x^{\alpha}}{x^{\alpha}-\beta} & \text { if }(\beta /(1-\beta))^{1 / \alpha} \leq x \leq 1 \\ x & \text { if } 1 \leq x \leq(\beta /(1-\beta))^{-1} .\end{cases}$
It is the fact that $f_{r}^{2}(x)=x$ and the right arm of the M-map being the square root of the identity over the interval $[1, G]$ that gives rise to a continuum of two-period cycles when $\tau=1$. This is of particular interest in that it allows a complete characterization of all the higher iterates of the M-map. ${ }^{10}$

The formulae look formidable, but the basic pattern is clear enough. Over the interval $[1, G]$, the $n$th iterate is the identity map for all even integers $n$, and is the restriction of the M-map itself to the same interval for odd $n$. More generally, the remaining interval [ 0,1 ], is divided into $n$ intervals with $n$ arms, and the basic observation is that for each subsequent iteration, $f^{-1}$ furnishes only one additional point that results in only the first arm splitting into two. The 5th and 6th iterates pictured in Fig. 3 give the necessary intuition. Note also that all points are periodic or eventually (not asymptotically) periodic. In terms of the language used in Matsuyama (1999), the Matsuyama system is not chaotic, and the fact that for the particular value $\tau=1$ under consideration, it is not difficult to demonstrate analytically that all cycles have the period length of 2 .

We can now present the general case. On defining $x_{n}$ as

$$
\begin{aligned}
& f_{l}^{n}\left(x_{n}\right)=1, \text { or equivalently } \\
& \quad x_{n}=G^{-\sum_{i=1}^{n}\left(\alpha^{i-1} / \alpha^{n}\right)}, \text { for } n=1,2, \ldots,
\end{aligned}
$$

we can assert the following.
Proposition 2. For $\tau=1$, and for $k=1,2, \ldots$, the even and odd iterates of the $M$-map are given by
$f^{2 k}(x)= \begin{cases}f_{l}^{2 k}(x) & \text { if } 0 \leq x \leq x_{2 k-1}, \\ \vdots & \vdots \\ f_{r}\left(f_{l}^{2 \ell+1}(x)\right) & \text { if } x_{2 \ell+1} \leq x \leq x_{2 \ell} \\ f_{l}^{2 \ell}(x) & \text { if } x_{2 \ell} \leq x \leq x_{2 \ell-1} \\ \vdots & \vdots \\ f_{r}\left(f_{l}(x)\right) & \text { if } x_{1} \leq x \leq 1 \\ x & \text { if } 1 \leq x \leq G,\end{cases}$

[^4]$f^{2 k+1}(x)= \begin{cases}f_{l}^{2 k+1}(x) & \text { if } 0 \leq x \leq x_{2 k}, \\ f_{r}\left(f_{l}^{2 k}(x)\right) & \text { if } x_{2 k} \leq x \leq x_{2 k-1} \\ \vdots & \vdots \\ f_{l}^{2 \ell+1}(x) & \text { if } x_{2 \ell+1} \leq x \leq x_{2 \ell} \\ f_{r}\left(f_{l}^{2 \ell}(x)\right) & \text { if } x_{2 \ell} \leq x \leq x_{2 \ell-1} \\ \vdots & \vdots \\ f_{l}(x) & \text { if } x_{1} \leq x \leq 1 \\ f_{r}(x) & \text { if } 1 \leq x \leq G,\end{cases}$
where $\ell$ takes value from $\{1,2, \ldots, k-1\}$.
Proof. In light of the earlier observations, the proof follows easily by induction. First, it is easy to verify that $f^{2 k}(\cdot)$ and $f^{2 k+1}(\cdot)$ are identical to $f^{2}(\cdot)$ and that $f^{3}(\cdot)$ when $k=1$. Using the fact that $f_{r}^{2}(\cdot)$ is an identity map, we can show that $f^{2 k+1}(x)=f\left(f^{2 k}(x)\right)$ and $f^{2(k+1)}(x)=f\left(f^{2 k+1}(x)\right)$.

We can now move beyond the second iterate.

## 3. The non-existence of a 3-period cycle

This tripartite section presents the third iterate, comments on the antecedent literature regarding the non-existence claim, and then offers its proof.

### 3.1. The third iterate of the M-map

In the determination of the third iterate, we stay with the basic procedure already delineated in the determination of the second iterate. Instead of the vertical through $(1, G)$, as in the determination of the second iterate, we focus on the critical benchmarks $x_{21}, x_{22}$ and $x_{23}$, as shown in Fig. 2, and constitutive elements of the set $\left(f^{2}\right)^{-1}(1)$. Unlike the case of the tent-map, for example, we do not obtain four points but only three: the procedure does not keep doubling the turning points because both arms of the M-map are not "onto". Thus, we obtain the seven intervals that go towards the determination of the third iterate. In any case, the procedure is now transparent; what is of relevance is that one can, in particular, chart out the turning points of as many iterates as we like, all on one map.

We present the algebraic specification of third iterate in Appendix, and turn the reader's attention to Fig. 2. Note that $f^{2}\left(x_{21}\right)=$ $f^{2}\left(x_{22}\right)=f^{2}\left(x_{23}\right)=1$, leading to $f^{3}\left(x_{21}\right)=f^{3}\left(x_{22}\right)=f^{3}\left(x_{23}\right)=G$. We also obtain $f^{3}\left(x_{11}\right)=\tau$ and $f^{3}\left(x_{12}\right)=\tau$. Finally, note that $f^{3}(1)=G \tau^{\alpha}$ and $f^{3}(G)=f\left(G \tau^{\alpha}\right)=\rho$, where the new parameter
$\rho \equiv \frac{\beta G^{2} \tau^{\alpha}}{\beta-1+G \tau^{\alpha}}$.
Remaining with Fig. 2, we see that the third iterate intersects the 45 -degree line only at the fixed point $\hat{x}$ of the M-map itself, and the two other fixed points of the second iterate designating a 2-period cycle, have disappeared. But of course, a rigorous proof of the non-existence of a 3-period cycle requires argumentation that goes beyond a picture. It requires a proof that the third iterate intersects the 45-degree line only at one point for all values in the admissible two-dimensional manifold of parameters, and that such a proof has not yet been offered in the antecedent literature since Matsuyama presented his model in 1999.

### 3.2. The claim and its antecedent literature

The following claim of obvious consequence for an understanding of the M-map and its attendant complexity is pervasive in the antecedent literature.

Theorem 1. There does not exist a 3-period cycle in the admissible parameter manifold of the M-map.

In this subsection, we ask whether there exists a formal proof of the above claim in the literature. We have already referred to Mitra's sufficient condition for Li-Yorke chaos, and his taking note of the following seminal footnote in Matsuyama (1999).

It is straightforward to show that this system is not chaotic in the sense of Li-York (sic), by demonstrating the non-existence of period-3 cycles. For this it suffices to show that $\Phi^{3}\left(k_{c}\right)>$ $k_{c}, \ldots$ Note, however that this does not rule out the possibility of chaotic trajectories. To rule out such a possibility, one needs to show that all the cycles have period length of a power of 2 , a property that is difficult to demonstrate analytically. ${ }^{11}$

The point is that the system is indeed chaotic in the sense of Li-Yorke, and that the (rather obvious) reason why the terse remarks in the footnote do not constitute a proof is simply that the Li-Yorke theorem requires that there exists a 3-period cycle starting from anywhere, and not necessarily one including the kink. It is this reason that makes Mitra's sufficient condition for Li-Yorke chaos all the more relevant for the M-map. As far as the literature since Matsuyama (1999) and Mitra (2001) is concerned, there is no mention of 3-period cycles in Mukherji (2005) except for a general remark on Li-Yorke chaotic maps; see Mukherji (2005, Footnote 10). In their section on "chaotic intervals", Gardini et al. (2008, Section 2.3), the authors write, "Indeed as we see in Fig. 2, it is also correct to say that cycles of period three cannot exist".

We now depict a piecewise-linear map in the left panel of Fig. 4. The map is defined as follows:
$f(x)=\left\{\begin{array}{cc}1.1+3.9 x & \text { if } x \leq 1 \\ 7.5-2.5 x & \text { if } \bar{x} \geq x>1 \\ \{(2.5 \bar{x}-7.5) /(5-\bar{x})\}(x-5) & \text { if } 5 \geq x \geq \bar{x}\end{array}\right.$
where $\bar{x}=2.95$. Its third iterate is depicted in the right panel of Fig. 4.

This map resembles the M-map in that the upward-sloping left arm is (weakly) concave and the downward-sloping right arm is convex. Although $f^{3}(1)>1$, and therefore there is no 3-period cycle starting from the critical point, it can be easily seen that there exist two 3-period cycles elsewhere on the map. ${ }^{12}$ This numerical example suggests that for a map with a concave left arm and a convex right arm, we cannot establish the non-existence of the 3 -period cycle by only investigating the critical point. ${ }^{13}$

### 3.3. A proof of the claim

The point then is that a rigorous analytical proof of the nonexistence of a 3-period cycle is needed. We provide such a proof in the remainder of this section based on the third iterate of the Mmap. Towards this end, we shall also need the inequalities collected as Lemma 1; their proof is relegated to Appendix and they can perhaps be profitably skipped by a reader interested only in the qualitative, rather than the quantitative, aspects of the M-map.

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Fig. 3. The fifth and sixth iterates of the M-map for $\tau=1$.


Fig. 4. A counter-example.

Lemma 1. For any $G$ in $(1,(1-\beta) / \beta)$, the following inequalities hold:
$\left(\max \left(\frac{(1-\beta)^{2}-\beta^{2} G^{3}}{(1-\beta-\beta G) G}, 0\right)\right)^{1 / \alpha}<x_{11}<\tau$.
We can now turn to the proof of Theorem 1 .
Proof. We begin the proof with the assertion that the interval $[\tau, G]$ constitutes an absorbing interval. ${ }^{14}$ Formally, we want to show that for any $x>0$, there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n>N, f^{n}(x) \in[\tau, G]$. First, $f([\tau, G])=[\tau, G]$, so if $x \in[\tau, G]$, then $f^{n}(x) \in[\tau, G]$ for any $n \in \mathbb{N}$. Second, consider $x \in(0, \tau)$. Note that $G x^{\alpha}>G x$. Pick $N$ to be the smallest integer that is greater than $(\ln \tau-\ln x) / \ln G$. Then $f_{l}^{N}(x)>\tau$, so $f^{n}(x) \in[\tau, G]$ for any $n \in \mathbb{N}$ with $n>N$. Last, the argument concerning $x \in(0, \tau)$ carries through for $x>G$ because $f(x) \in(0, \tau)$.

A 3-period cycle, if it exists, must be on the absorbing interval [ $\tau, G]$. First, at least one periodic point is on the left arm of the M-map. If this is not the case, then according to the 3rd iterate of

[^6]the M-map (also see the 6th arm in Fig. 2), the following equation must admit at least three distinct roots:
$x=\frac{\beta^{3} G^{3} x}{(\beta-1)^{3}+\left[(\beta-1)^{2}+(\beta-1) \beta G+\beta^{2} G^{2}\right] x}$.
There is a unique solution to the equation above, which is the fixed point of $f$, so a 3-period cycle cannot occur entirely on the right arm of the M-map. Second, at most one point of a 3-period cycle must be on the left arm of the M-map. If this is not the case, two periodic points on the left arm must occur consecutively in a 3-period cycle. However, since we know $G \tau^{\alpha}>1$, which implies $G x^{\alpha}>1$ for any $x \in[\tau, 1]$, this leads to a contradiction.

We have now shown that there must be one point on the left arm and two points on the right arm in a three-period cycle. According to our formula of the third iterate of the M-map (also see the 3rd arm in Fig. 2), this suggests that there exists $\hat{x} \in[\tau, 1)$ such that $f_{r}^{2}\left(f_{\ell}(\hat{x})\right)=\hat{x}$, or equivalently,
$\frac{\beta^{2} G^{3} \hat{x}^{\alpha}}{(\beta-1)^{2}+(\beta-1+\beta G) G \hat{x}^{\alpha}}=\hat{x}$.

Since

$$
\begin{aligned}
& \frac{\beta^{2} G^{3} x^{\alpha}}{(\beta-1)^{2}+(\beta-1+\beta G) G x^{\alpha}}>x^{\alpha} \\
& \text { if } x>\left(\max \left(\frac{(1-\beta)^{2}-\beta^{2} G^{3}}{(1-\beta-\beta G) G}, 0\right)\right)^{1 / \alpha}
\end{aligned}
$$

according to Lemma 1 , this implies
$\frac{\beta^{2} G^{3} x^{\alpha}}{(\beta-1)^{2}+(\beta-1+\beta G) G x^{\alpha}}>x^{\alpha}$ for $x \geq \tau$,
which further implies that $\hat{x}>\hat{x}^{\alpha}$. However, $\chi^{\alpha}>x$ for $x<1$, and we obtain the contradiction and complete the proof.
It bears emphasis that two of the steps used in the proof presented above are available in the literature. First, the claim that the interval $[\tau, G]$ is an absorbing interval constitutes Lemma 6.1 in Yano et al. (2011). ${ }^{15}$ Second, the proof relies on Lemma 1 whose proof is relegated to the Appendix, and where we explicitly note that only one of the inequalities of the lemma is available as Footnote 8 in Matsuyama (1999). However, the proof presented above relies on both inequalities.

Next, we move on to the fourth iterate, and to the possibility of a 5-period cycle.

## 4. The non-existence of a 5 -period cycle

The fact that the M-map is not complicated enough to have a 3-period cycle anywhere in its manifold of admissible parameters is a folk-result in the sense that it is well-known, even if it was not so far proved. What is not well-known, and to the knowledge of the authors never asked, is whether the M-map is complicated enough to have a 5-period cycle anywhere in its manifold of admissible parameters. We turn to this question in this section.

### 4.1. The fourth iterate of the M-map

Consider Fig. 5 and the fourth iterate of the M-map. As mentioned in the opening paragraph of Section 3, the procedure to determine it is by now quite routine. Instead of the set $\left(f^{2}\right)^{-1}(1)$, we work with $\left(f^{3}\right)^{-1}(1)$ to obtain the five points $x_{3 i}, i=1, \ldots, 5$, and, in general, the twelve intervals. The qualifier "in general" highlights the fact that there may be only 11 arms to the fourth iterate depending on the parameter values, as depicted in Fig. 5. All this is a continuing testament to the fact that unlike the tent-map, one of the two arms of the M-map is not "onto" the unit interval. ${ }^{16}$

Again, we spare the reader the detailed algebraic specification of the fourth iterate by relegating it to Appendix, and focusing his/her attention on its diagrammatic representation in Fig. 5. We see that the fourth iterate intersects the 45 -degree line at three points: the fixed point $\hat{x}$ of the M-map itself and the unique 2-period cycle. These findings can be usefully compared with Figures 4 in Gardini et al. (2008). ${ }^{17}$ In looking at Fig. 5, note that in line with the iterative procedure we are following, $f^{3}\left(x_{3 i}\right)=1, i=$ $1, \ldots, 5$, necessarily implies $f^{4}\left(x_{3 i}\right)=G, i=1, \ldots, 5$. Note also that
$f^{4}\left(x_{21}\right)=f^{4}\left(x_{22}\right)=f^{4}\left(x_{23}\right)=\tau$ and $f^{4}\left(x_{11}\right)=f^{4}\left(x_{12}\right)=G \tau^{\alpha}$,

[^7]

Fig. 5. The fourth iterate of the M-map for $\tau<1$ and $\rho<1$.
that $f^{4}(1)=\rho$ and that $f^{4}(G)=f(\rho)$. Finally, note that there exists the 12th arm if and only if $G>x_{35}$, or equivalently, $\tau<x_{22}$, which is also equivalent to $\rho>1$, and that the 13th arm does not exist because we have already shown that $\tau>x_{11}$.

### 4.2. The result and its proof

The surprise here is that to show the non-existence of a 5-period cycle, we do not need to compute the fifth iterate and to show that it intersects the 45-degree line only at one point for all values in the admissible two-dimensional manifold of parameters. The algebraic specification of the fourth iterate suffices! We begin with a statement of the second principal claim of the paper.

Theorem 2. There does not exist a 5-period cycle in the admissible parameter manifold of the M-map.

The proof of the result relies on the following inequalities, and we again relegate their proofs to Appendix.

Lemma 2. For any Gin $(1,(1-\beta) / \beta)$, the following inequalities hold:
(i) $G \tau^{\alpha}>x_{23}$, (ii) $\rho<x_{12}$,
(iii) $\left(\max \left(\frac{(\beta-1)^{4}-\beta^{4} G^{5}}{G(1-\beta-\beta G)\left[(\beta-1)^{2}+\beta^{2} G^{2}\right]}, 0\right)\right)^{1 / \alpha}<x_{33}$.

We can now turn to the proof of the theorem.
Proof. First, a 5-period cycle can only occur over the range $[\tau, G]$. According to Lemma 1 , this implies that a fixed point of $f^{5}(\cdot)$ must be no less than $x_{11}$. Second, if there is a five-period cycle, at least one point on the cycle must be on the left arm. This can be seen by solving the fixed point for the fifth-iterate of the right arm given by
$\frac{\beta^{5} G^{5} x}{(\beta-1)^{5}+\left[(\beta-1)^{4}+\beta G(\beta-1)^{3}+\beta^{2} G^{2}(\beta-1)^{2}+\beta^{3} G^{3}(\beta-1)+\beta^{4} G^{4}\right] x}$ $=x$.
There is a unique fixed point and it coincides with the fixed point of $f(\cdot)$, so it is impossible to have a five-period cycle with all the points on the right arm of the original M-map. Hence, for there to be a five-period cycle, there must exist a fixed point of $f^{5}(\cdot)$ in $\left[x_{11}, 1\right]$.

We first consider the interval $\left[x_{11}, x_{33}\right]$. The fourth iterate on this interval increases with $x$, and we have $f^{4}\left(x_{11}\right)=G \tau^{\alpha}>1$ and $f^{4}\left(x_{33}\right)=G$. This implies that the fifth iterate on this interval decreases with $x$. According to Lemma 2(i), $G \tau^{\alpha}>x_{23}$, so $f^{5}\left(x_{33}\right)=$ $\tau>\left(x_{23} / G\right)^{1 / \alpha}=x_{33}$. Therefore, $f^{5}(x) \geq f^{5}\left(x_{33}\right)>x_{33} \geq x$ for $x \in\left[x_{11}, x_{33}\right]$. There is no fixed point of the fifth-iterate on this interval.

Next, consider the interval [ $x_{33}, x_{22}$ ]. The fourth iterate on this interval decreases with $x$, and we have $f^{4}\left(x_{33}\right)=G$ and $f^{4}\left(x_{22}\right)=$ $\tau<1$. Define $x_{4} \in\left[x_{33}, x_{22}\right]$ such that $f^{4}\left(x_{4}\right)=1$. Then the fifth iterate increases with $x$ for $x \in\left[x_{33}, x_{4}\right]$. It can be shown that

$$
\begin{aligned}
& \frac{\beta^{4} G^{5} \chi^{\alpha}}{(\beta-1)^{4}+\left[(\beta-1)^{3}+\beta G(\beta-1)^{2}+\beta^{2} G^{2}(\beta-1)+\beta^{3} G^{3}\right] G x^{\alpha}}>x^{\alpha} \\
& \Longleftrightarrow x>\left(\max \left(\frac{(\beta-1)^{4}-\beta^{4} G^{5}}{G(1-\beta-\beta G)\left[(\beta-1)^{2}+\beta^{2} G^{2}\right]}, 0\right)\right)^{1 / \alpha}
\end{aligned}
$$

According to Lemma 2(iii), this implies the inequality holds for $x \geq x_{33}$. Moreover, $x^{\alpha}>x$ for $x \leq x_{4}<1$. Combining these two inequalities, we have $f^{5}(x)>x$ for $x \in\left[x_{33}, x_{4}\right]$. The fifth iterate decreases with $x$ for $x \in\left[x_{4}, x_{22}\right]$. We have $f^{5}(x) \geq f^{5}\left(x_{22}\right)=$ $G \tau^{\alpha}>1>x_{22} \geq x$. Therefore, there is no fixed point in $\left[x_{33}, x_{22}\right]$.

Consider the last interval $\left[x_{22}, 1\right]$. The fourth iterate on this interval increases with $x$. There are two possible cases. If $f^{4}(1)=$ $\rho \leq 1$, the fifth arm increases with $x$ on this interval, so $f^{5}(x) \geq$ $f^{5}\left(x_{22}\right)>1 \geq x$. If $f^{4}(1)>1$, the fifth arm first increases and then decreases with $x$ for $x \in\left[x_{22}, 1\right]$. However, as Lemma 2(ii) implies $\rho<(1-\beta) /(1-G \beta)$ or equivalently $f(\rho)=(G \beta \rho) /(\beta-1+\rho)>1$, we again have $f^{5}(x)>1 \geq x$ for any $x \in\left[x_{22}, 1\right]$. There is no fixed point in $\left[x_{22}, 1\right]$ and the proof is complete.

## 5. The existence of cycles of 6- and higher periods

As emphasized in Section 1, the motivation of this work is to shift the emphasis from numerical determinations of the existence of "chaos" to an analytical examination of how rich and complicated are the dynamics that one can associate with the Mmap. We have also been emphasizing that our primary concern is with questions of existence rather than that of the stability or the genericity of periodic orbits. As such, we ask for the smallest value of odd $n>5$ for which $n$-period cycles, stable or unstable, exist?

### 5.1. A 7-period cycle

A numerical answer is furnished in the following example that presents a trajectory of a 7-period cycle that includes the critical point (kink) of the M-map.

## Example 1.

For $\alpha=0.99$, there exists $G^{*} \in(1,(1-\beta) / \beta)\left(G^{*} \approx 1.01016\right)$ such that a 7-period cycle starts from the critical point of the M-map.

The example is diagrammed in Fig. 6. It is also of some interest that we can underscore our analytical findings regarding the fourth iterate by an example within the same parametric regime given by $\alpha=0.99$.

## Example 2.

For $\alpha=0.99$, there exists $G^{* *} \in(1,(1-\beta) / \beta)\left(G^{* *} \approx 1.29886\right)$ such that a 4 -period cycle starts from the critical point of the M-map.

These examples, and especially the first, are of interest in their own right, but we now use them instrumentally to present a result in the following subsection: it completes the existence question that we pose in this paper, and can be usefully compared with


Fig. 6. Example 1 on the existence of a 7-period cycle.

Claim 12 and Proposition 1 in Mukherji (2005). Let us also take this opportunity to draw attention to the fact that Sharkovsky's theorem guarantees that in this parametric regime ( $\alpha=0.99$ ), there exist cycles of all periods greater than or equal to seven. ${ }^{18}$

### 5.2. A general result

Consider an $n$-period cycle with its orbit specified in the following way: it starts from the critical point 1, and, after hitting $G$ and $G \tau^{\alpha}$, it stays on the right arm of the M-map until it hits the critical point again. Fig. 6 illustrates an example of this type of cycle. Formally, there exists an $n$-period ( $n \geq 3$ ) cycle with this particular orbit if $f_{r}^{n-3}\left[f_{l}\left(f_{r}^{2}(1)\right)\right]=1$, or equivalently,

$$
\begin{aligned}
G \tau^{\alpha} & =\left[\frac{1}{1+G \beta-\beta}+\left(\frac{G \beta}{\beta-1}\right)^{n-4} \frac{G \beta^{2}(1-G)}{(1-\beta)(1+G \beta-\beta)}\right]^{-1} \\
& \equiv \varphi_{n}(\beta, G)
\end{aligned}
$$

We can now consider a subset of the 2-dimensional parameter space given by
$\mathcal{P}_{n}=\left\{(\beta, G) \in(1 / e, 1 / 2) \times(1,(1-\beta) / \beta): G \tau^{\alpha}=\varphi_{n}(\beta, G)\right\}$.
Then Theorems 1 and 2 imply that $\mathcal{P}_{n}=\emptyset$ for $n=3,5$. The question then pertains to higher values of $n$. We can use Examples 1 and 2 to present the following consequence of the intermediate value theorem.

Proposition 3. For $n \geq 3, \mathcal{P}_{n} \neq \emptyset$ for all $n \neq 3$ and 5 .
Proof. For $G \in(1,(1-\beta) / \beta), \varphi_{n}(\beta, G) \equiv y_{n} \in\left(y_{7}, y_{4}\right)$ for any $n \geq 6$ and $n \neq 7$. In Examples 1 and 2, we have shown that for $\alpha=0.99, G^{*} \tau^{\alpha}=y_{7}$ and $G^{* *} \tau^{\alpha}=y_{4}$. Therefore, when $\alpha=0.99$, $n \geq 6$, and $n \neq 7$, we have $y_{n}>y_{7}=G \tau^{\alpha}$ for $G=G^{*}$ and $y_{n}<y_{4}=G \tau^{\alpha}$ for $G=G^{* *}$. When $\alpha$ is given, both $G \tau^{\alpha}$ and $y_{n}$ are continuous in $G$. According to the intermediate value theorem, for $\alpha=0.99$, there exists $G \in\left(G^{* *}, G^{*}\right)$ such that $y_{n}=G \tau^{\alpha}$ for $n \geq 6$ and $n \neq 7$. Therefore, $\mathcal{P}_{n} \neq \emptyset$ for all $n=4$ or $n \geq 6$. This completes our proof.

[^8]Our proof is based on the numerical experiments with $\alpha=$ 0.99 . Indeed, $\alpha$ has to be sufficiently close to one for $\mathcal{P}_{n}$ to be nonempty ( $n \neq 3$ and 5 ).

However useful numerical examples prove to be, they surely need underscoring by an analytical result. A natural question concerns parameter values under which 7-period cycles exists! We answer this question by appealing to the generalization of the LiYorke theorem presented in Li et al. (1982), henceforth LMPY, and further refined by Ruette (2017, Proposition 3.34). We can present

Theorem 3. A sufficient condition for the existence of a 7-period cycle of the M-map is given by

$$
\begin{aligned}
0 & <f_{r}^{3}(\rho) \\
& =\frac{\beta^{4} G^{5} \tau^{\alpha}}{(\beta-1)^{4}+\left[(\beta-1)^{3}+\beta G(\beta-1)^{2}+\beta^{2} G^{2}(\beta-1)+\beta^{3} G^{3}\right] G \tau^{\alpha}} \leq 1 .
\end{aligned}
$$

Proof. According to the LMPY theorem, there exists a 7-period cycle if $f^{7}(x) \leq x<f(x)$ for some $x$. In applying this result, we work with $x=1$. Since $f(1)=G>1$, for a 7 -period cycle, we need conditions that guarantee $f^{7}(1) \leq 1$. We have already shown that $f^{2}(1)=\tau<1$ and that $f^{3}(1)=G \tau^{\alpha}>1$. We now want to show $f^{4}(1)=f_{r}\left(G \tau^{\alpha}\right) \equiv \rho>1, f_{r}(\rho)>1$, and $f_{r}^{2}(\rho)>1$.

First, $f_{r}(\rho)>1$ directly follows Lemma 2(i). Second, $f_{r}^{3}(\rho) \in$ $(0,1]$ implies
$0<\left[\frac{\beta G-\beta}{1+\beta G-\beta}\left(\frac{\beta G}{\beta-1}\right)^{4}+\frac{1}{1+\beta G-\beta}\right] G \tau^{\alpha} \leq 1$,
which further implies $G \tau^{\alpha}<1+\beta G-\beta$.
By construction, $f^{3}(1)>1$ implies $\rho>0$ and $f_{r}(\rho)>1$ implies $f_{r}^{2}(\rho)>0$. If $\rho \in(0,1]$, then $G \tau^{\alpha} \geq(1-\beta) /(1-\beta G)>1+\beta G-\beta$, contradicting the inequality we have obtained above. If $f_{r}^{2}(\rho) \in$ ( 0,1 ], we have

$$
\left[\frac{\beta G-\beta}{1+\beta G-\beta}\left(\frac{\beta G}{\beta-1}\right)^{3}+\frac{1}{1+\beta G-\beta}\right] G \tau^{\alpha} \geq 1
$$

Since $\beta<1$, this implies $G \tau^{\alpha}>1+\beta G-\beta$, which again leads to contradiction.

In sum, we have shown $f^{4}(1)=\rho>1, f_{r}(\rho)>1, f_{r}^{2}(\rho)>1$. This implies that $f_{r}(\rho)=f^{5}(1)>1, f_{r}^{2}(\rho)=f^{6}(1)>1, f_{r}^{3}(\rho)=$ $f^{7}(1)$, and therefore, $f^{7}(1) \leq 1$. Applying the LMPY theorem, we obtain a 7 -period cycle.

We now conclude this section with

## Example 3.

For $\alpha=0.99$, there exists $G=1.01015$ such that the inequality identified in Theorem 3 holds strictly.

Moreover, when $\alpha=0.99$ and $G=1.01015$, the conditions identified in Theorem 6.1 of Yano et al. (2011) are also satisfied. This suggests that for appropriate parameter values, the M-map exhibits a special form of ergodic chaos with cycles of all periods but 3 or $5 .{ }^{19}$

## 6. Concluding remark

The points this paper makes, and the results it reports, are sharp enough that one need not go beyond the abstract and the introduction so as to summarize them yet again. Instead, we conclude with a simple epistemological remark.

Since May's influential 1976 piece in Nature emphasizing complicated dynamics from simple models, there has been a numerical

[^9]turn in that questions of comparative economic dynamics, and therefore of economic dynamics, depend essentially of identification of parametric ranges within which the map does not change abruptly. This is not true for the earlier literature, both in growth and development, as well as in macroeconomics, in which broad qualitative assumptions such as convexity or super-modularity sufficed. Iterates are now sure to play an important and obvious part in further elaborations that discipline ad hoc savings function by optimizing behavior, as for example, in Matsuyama (2001), Dudek (2010), Kennedy and Stockman (2008), Stockman (2010), and by adducing additional stylized facts and temporal specifications stemming from short- and the long-run. Much more remains to be done.

## Appendix

We begin with the proofs of the lemmata.
Proof of Lemma 1. The second assertion has been proved in Matsuyama (1999), and we transcribe the proof there under our notation here. Define $h_{1}(G) \equiv \beta G^{2+1 / \alpha}-(G+\beta-1)$, and note that $h_{1}(1)=0$, and $h_{1}^{\prime}(G)=(2+1 / \alpha) \beta G^{1+1 / \alpha}-1>0$ for $G>1$, because $2+1 / \alpha>3$ and $\beta>1 / e>1 / 3$. This implies $h_{1}(G)>0$ for $G>1$, and so $\tau-x_{11}=h_{1}(G) /\left[G^{1 / \alpha}(G+\beta-1)\right]>0$ for any $G>1$.

For the first assertion, if $(1-\beta)^{2} \leq \beta^{2} G^{3}$, then the inequality trivially holds. If $(1-\beta)^{2}>\beta^{2} G^{3}$, it suffices to show that

$$
\frac{(1-\beta)^{2}-\beta^{2} G^{3}}{(1-\beta-\beta G)}<1 \Longleftrightarrow \beta G-\beta^{2} G^{3}-\beta+\beta^{2}<0
$$

Define $h_{2}(G)=\beta G-\beta^{2} G^{3}-\beta+\beta^{2}$, and note that $h_{2}(1)=0$ and $h_{2}^{\prime}(G)=\beta-3 \beta^{2} G^{2}<0$ for $G>1$. Hence, $h_{2}(G)<0$ for any $G>1$.

Proof of Lemma 2. Since $\tau<1$, it suffices to show for (i) that

$$
\begin{aligned}
G \tau & =\frac{\beta G^{3}}{\beta-1+G}>\frac{(1-\beta)^{2}}{G^{2} \beta^{2}+(1-\beta-\beta G)} \\
& \Longleftrightarrow \beta G^{3}\left(\beta^{2} G^{2}+1-\beta-\beta G\right)>(1-\beta)^{2}(\beta-1+G)
\end{aligned}
$$

Define $h_{3}(G) \equiv \beta G^{3}\left(\beta^{2} G^{2}+1-\beta-\beta G\right)-(1-\beta)^{2}(\beta-1+G)$. We want to show $h_{3}(G)>0$ for $G>1$. Since $h_{3}(1)=0$, we just need to show
$h_{3}^{\prime}(G)=5 \beta^{3} G^{4}-4 \beta^{2} G^{3}+3(1-\beta) \beta G^{2}-(1-\beta)^{2}>0$ for $G>1$.
Define $\ell(\beta) \equiv h_{3}^{\prime}(1)=5 \beta^{3}-4 \beta^{2}+3(1-\beta) \beta-(1-\beta)^{2}$. Simple calculation shows $\ell(1 / e) \approx 0.006>0$, and $\ell^{\prime}(\beta)=$ $15 \beta^{2}-16 \beta+5>0$ for $\beta \in(1 / e, 1 / 2)$, so $h_{3}^{\prime}(1)>0$. Consider the second derivative $h_{3}^{\prime \prime}(G)=20 \beta^{3} G^{3}-12 \beta^{2} G^{2}+6(1-\beta) \beta G=$ $2 \beta G\left[10 \beta^{2} G^{2}-6 \beta G+3(1-\beta)\right]$. It is straightforward to show that $h_{4}(G) \equiv 10 \beta^{2} G^{2}-6 \beta G+3(1-\beta)$ is an increasing function of $G$ and $h_{4}(1)>0$, so $h_{3}^{\prime \prime}(G)>0$. This implies $h_{3}^{\prime}(G)>0$ for $G>1$.

For (ii), we want to show explicitly that

$$
\begin{aligned}
\tau^{\alpha} & >\frac{(1-\beta)^{2}}{(1-\beta) G-(1-\beta G) \beta G^{2}} \text { or that } \tau \\
& >\frac{(1-\beta)^{2}}{(1-\beta) G-(1-\beta G) \beta G^{2}},
\end{aligned}
$$

since $\tau<1$. This reduces to the case to (i) above.
For (iii), if $(\beta-1)^{4}-\beta^{4} G^{5} \leq 0$, then the inequality trivially holds. We then consider $(\beta-1)^{4}-\beta^{4} G^{5}>0$. Rearranging the inequality, we want to show
$\frac{(\beta-1)^{4}-\beta^{4} G^{5}}{(1-\beta-\beta G)\left[(\beta-1)^{2}+\beta^{2} G^{2}\right]}<\frac{(1-\beta)^{2}}{\beta^{2} G^{2}+(1-\beta-\beta G)}$,

$$
f^{4}(x)= \begin{cases}G^{1+\alpha+\alpha^{2}+\alpha^{3}}(x)^{\alpha^{4}} & \text { if } 0 \leq x \leq x_{31} \equiv G^{-\left(\alpha^{2}+\alpha+1\right) / \alpha^{3}}, \\ \frac{\beta G^{\alpha^{2}+\alpha+2} \chi^{\alpha^{3}}}{\beta-1+G^{\alpha^{2}+\alpha+1} x^{\alpha^{3}}} & \text { if } x_{31} \leq x \leq x_{21} \\ \frac{\beta^{2} G^{\alpha+3} x^{\alpha^{2}}}{(\beta-1)^{2}+(\beta-1+\beta G) G^{\alpha+1} x^{\alpha^{2}}} & \text { if } x_{21} \leq x \leq x_{32} \equiv\left(\frac{1-\beta}{(1-\beta G) G^{\alpha+1}}\right)^{1 / \alpha^{2}} \\ G\left(\frac{\beta G^{\alpha+2} x^{\alpha^{2}}}{\beta-1+G^{\alpha+1} \chi^{\alpha^{2}}}\right)^{\alpha} & \text { if } x_{32} \leq x \leq x_{11} \\ G\left(\frac{\beta^{2} G^{3} x^{\alpha}}{(\beta-1)^{2}+(\beta-1+\beta G) G x^{\alpha}}\right)^{\alpha} & \text { if } x_{11} \leq x \leq x_{33} \equiv\left(\frac{(1-\beta)^{2}}{\beta^{2} G^{3}+(1-\beta-\beta G) G}\right)^{1 / \alpha} \\ \frac{\beta^{3} G^{4} x^{\alpha}}{(\beta-1)(\beta-1+G)^{3}+\left[(\beta-1)^{2}+(\beta-1) \beta G+\beta^{2} G^{2}\right] G x^{\alpha}} & \text { if } x_{33} \leq x \leq x_{22} \\ \frac{\beta^{1+\alpha} G^{2+2 \alpha} x^{\alpha^{2}}+\beta^{\alpha} G^{1+2 \alpha} x^{\alpha^{2}}}{(\beta-1)\left[(\beta-1)^{2}+(\beta G+\beta-1) x\right]^{\alpha}+\beta^{2 \alpha} G^{1+2 \alpha} X^{\alpha}} & \text { if } x_{22} \leq x \leq 1 \\ \frac{\beta^{4} G^{4} x}{(\beta-1)^{4}+\left[(\beta-1)^{3}+\beta G(\beta-1)^{2}+\beta^{2} G^{2}(\beta-1)+\beta^{3} G^{3}\right] x} & \text { if } x_{23} \leq x \leq x_{34} \equiv \frac{(1-\beta)^{3}}{(1-\beta)^{2}-\beta^{3} G^{3}-(1-\beta-\beta G) \beta G} \\ G\left(\frac{\beta^{3} G^{3} x}{(\beta-1)^{3}+\left[(\beta-1)^{2}+(\beta-1) \beta G+\beta^{2} G^{2}\right] x}\right)^{\alpha} & \text { if } x_{34} \leq x \leq x_{12} \\ G\left(\frac{\beta^{1+\alpha} G^{2+\alpha} x^{\alpha}}{(\beta-1)(\beta-1+x)^{\alpha}+\beta^{\alpha} G^{1+\alpha} x^{\alpha}}\right)^{\alpha} & \text { if } x_{12} \leq x \leq \min \left\{G, x_{35} \equiv \frac{(1-\beta) x_{22}}{x_{22}-\beta G}\right\} \\ \frac{\beta^{2+\alpha} G^{3+\alpha} x^{\alpha}}{(\beta-1)^{2}(\beta-1+x)^{\alpha}+(\beta-1+\beta G) \beta^{\alpha} G^{1+\alpha} x^{\alpha}} & \text { if } x_{35} \leq x \leq G \text { and } x_{35}<G .\end{cases}
$$

Box I.
or equivalently,

$$
\begin{aligned}
& (\beta-1)^{4} \beta^{2} G^{2}-\beta^{6} G^{7}-(1-\beta-\beta G) \beta^{4} G^{5} \\
& \quad<(1-\beta-\beta G)(1-\beta)^{2} \beta^{2} G^{2} .
\end{aligned}
$$

This inequality can be further simplified to $\beta G^{3}\left(\beta^{2} G^{2}+1-\beta-\right.$ $\beta G)>(1-\beta)^{2}(G+\beta-1)$, which has been established above.

The third iterate of $f(\cdot)$ is given by $f^{3}(x)$

$$
\begin{aligned}
& \left\{\begin{array}{l}
G^{1+\alpha+\alpha^{2}}(x)^{\alpha^{3}} \\
\frac{\beta G^{2+\alpha} x^{\alpha^{2}}}{\beta-1+G^{\alpha+1} \alpha^{\alpha^{2}}} \\
\frac{\beta^{2} G^{3} x^{\alpha}}{(\beta-1)^{2}+(\beta-1+\beta G) G X^{\alpha}}
\end{array}\right. \\
& \text { if } 0 \leq x \leq x_{21} \\
& \equiv G^{-(\alpha+1) / \alpha^{2}} \text {, } \\
& \text { if } x_{21} \leq x \leq x_{11} \\
& \text { if } x_{11} \leq x \leq x_{22} \\
& \equiv\left(\frac{1-\beta}{G-\beta G^{2}}\right)^{1 / \alpha} \\
& =\left\{\begin{array}{l}
G\left(\frac{\beta G^{2} x^{\alpha}}{\beta-1+G x^{\alpha}}\right)^{\alpha} \\
G\left(\frac{\beta^{2} G^{2} x}{(\beta-1)^{2}+(\beta-1+\beta G) x}\right)^{\alpha}
\end{array}\right. \\
& \text { if } x_{22} \leq x \leq 1 \\
& \text { if } 1 \leq x \leq x_{23} \\
& \equiv \frac{(1-\beta)^{2}}{\beta^{2} G^{2}+(1-\beta-\beta G)} \\
& \begin{cases}\frac{\beta^{3} G^{3} x}{(\beta-1)^{3}+\left[(\beta-1)^{2}+(\beta-1) \beta G+\beta^{2} G^{2}\right] x} & \text { if } x_{23} \leq x \leq x_{12} \\
\frac{\beta^{1+\alpha} G^{2+\alpha} \chi^{\alpha}}{(\beta-1)(\beta-1+x)^{\alpha}+\beta^{\alpha} G^{1+\alpha} \chi^{\alpha}} & \text { if } x_{12} \leq x \leq G .\end{cases}
\end{aligned}
$$

Next, the fourth $f^{4}(\cdot)$ iterate of $f(\cdot)$ is given in Box I.

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[^0]:    The authors gratefully acknowledge the comments and suggestions of the Chief Editor and three anonymous referees that led to a substantive reconfiguration of the initial submission. They are especially grateful to Laura Gardini for her generous reading, and many suggestions and directions for further work that we hope to take up in the future. The authors thank Y. Chen, J. Quah, D. Ray, R. Serrano and R. Vohra for sage advice and encouragement to aim higher. The authors also thank A. Piazza for showing them the quicker route to the iterations reported here; C. Avram, S. Siddiqui and K. Reffett for additional pointers concerning computing; and M. Fujio, O. Khan, M. Linask, and M. Uyanik for ongoing collaboration and correspondence. Some of the results reported here were presented at the APET-16 session held in Rio de Janeiro on August 12, 2016, titled "Income Contingent Loans and the Economics of Education" and chaired by Bruce Chapman. Versions of the results reported here were presented by Khan at the Johns Hopkins Theory Seminar April 16, 2017; at the Kansas Workshop in Economic Theory May 5, 2017; and at the XXVI European Workshop on General Equilibrium Theory June 1, 2017.

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    1 Ruette continues, "It relies generally on the idea of unpredictability or instability, i.e. knowing the trajectory is not enough to know what happens elsewhere." The sentiment expressed in these sentences is part of the folklore of the subject; we cull them from Ruette (2017, p. iv).

[^1]:    2 The adjectives "dynamic and "stagnant" bow to the conventional categorization: in the Solow regime, the economy is growing through capital accumulation at an exogenous rate of population growth, and in the Romer regime, through the expansion of product varieties at an endogenously-generated growth rate.
    3 See Fig. 1. Econometricians will note and appreciate the distinction between the standard logistic function and the formula for the right arm given in Eq. (1). The former is given by $1 /(a+b \exp (-x))$, both $a$ and $b$ positive, but a change-of-variable in the M-map leads to at least one of the two parameters being negative; see for example deCani and Stine (1986) and their reference to the text of Johnson-Kotz, now revised as Johnson-Kotz-Balakrishnan. Also see Robinson's text (Robinson, 1995 p. 2).

[^2]:    4 To be sure, this is our subjective, and perhaps opinionated reading of Mitra (2001); only an individual author knows his or her motivation.

    5 The uninitiated reader should note that all these papers involve existence of the relevant cycle, and that their concern is not with stability.
    6 For this literature, see Gancia and Zilibotti (2005), Dudek (2010), Matsuyama et al. (2016), Yano and Furukawa (2013) and their references.

[^3]:    7 As we observe below, this case has especially caught the attention of the growth and development literature; see Gancia and Zilibotti (2005) and Yano and Furukawa (2013).
    8 It is important, given Mitra and Nishimura (2001), that the reader not give more than the necessary weight to our claim: Yano et al. (2011), and subsequent work in Sato and Yano (2012a, b, 2013) and Yano et al. (2013), work with the second iterate, and Mukherji (2005)[t] and Gardini et al. (2008) diagram numerical snapshots of both the 2nd and 4th iterates. Our point simply is that explicit formulae for these iterates of the M-map, as are offered below, have not been furnished and used for analytical results.
    9 In addition to Matsuyama (1999) see, for example, Matsuyama (2001), Mukherji (2005), Gancia and Zilibotti (2005) and Yano et al. (2013, Appendix).

[^4]:    10 For an exposition of Stefan's 1977 construction of the square root of a map, see Ruette (2017, Example 3.22).

[^5]:    11 See Matsuyama (1999, Footnote 8). The footnote is an important marker of the professional understanding of dynamical systems at the time. It continues, "Another difficulty is that the Schwartzian derivative of the map, $\Phi$, is not negative, which means, among other things, that the iteration of the critical point, $\Phi^{\prime}\left(k_{c}\right)$, may fail to detect stable cycles, even if they exist". Stability of the cycles is not our (or LiYorke's) concern in this essay.
    12 Indeed, one of the 3-period cycles is stable, though stability is not the concern here.
    13 This observation is perhaps more pertinent to a map more general than the specific M-map.

[^6]:    14 This absorbing interval has been established by Mukherji (2005). We thank one of our referees for pointing this out.

[^7]:    15 The authors do not write out the straightforward proof, and we do so only for completeness and for the reader's convenience. We also take this opportunity to thank two anonymous referees for seeing the need to couch our arguments in terms of the absorbing interval.
    16 In this, the M-map shares a commonality with the check-map studied in, for example, Khan and Mitra $(2005,2013)$ and Khan and Piazza $(2011)$.
    17 In this comparison, note the fact that there is no continuum of 4-period cycles as analytically revealed by the algebraic specification of the fourth iterate. We exaggerate this comparison by giving more curvature to the representation of the iterate in the interval ranges $\left[x_{22}, 1\right]$.

[^8]:    18 See, for example, Khan and Piazza (2011, p. 413) for a discussion of the theorem in the context of the RSS model and the checkmap.

[^9]:    19 We thank one of our referees for pointing out this connection.

