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# Exact parametric restrictions for 3-cycles in the RSS model: A complete and comprehensive characterization[☆]

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ABSTRACT

This paper presents necessary and sufficient conditions for 3-period cycles in the two-sector Robinson–Solow–Srinivasan (RSS) model, taking as its point of departure an independently-(and simultaneously-) discovered exact discount-factor restriction for a general class of growth models by Mitra and Nishimura–Yano (MNY) in 1996. Our investigation of this remarkable result in the specificity of the RSS model enables a broadened inquiry that goes beyond the discount factor to parameters of labor-productivity and capital-depreciation. Since the RSS model, despite its concrete simplicity, is not covered by the general MNY model, the exact discount-factor restriction presented here does not follow from the MNY theorem, and necessitates new argumentation. Furthermore, we present a novel *exact* parametric region as our second result.



*A violent order is a disorder; and a great disorder is an order. These two things are one.* Wallace Stevens (1942)[1]

*The unassimilable fact leads us on: we are led on to the boundaries where relations loosen into chaos.* A. R. Ammons (1961)[2]

## 1. Introduction

In an instance of independent and simultaneous discovery of 24 years ago, Mitra (1996) and Nishimura and Yano (1996) presented the following result.[3]

**Theorem MNY.** *For a class $\mathfrak{G}$ of qualitatively-delineated dynamic optimization models, there exists $\bar{\rho} = [(\sqrt{5}-1)/2]^2$ with the following two properties:*

*(i) if the optimal policy function of any $G \in \mathfrak{G}$ exhibits a 3-period cycle, then the discount factor necessarily lies in $(0, \bar{\rho})$;*

*(ii) for any discount factor in $(0, \bar{\rho})$, there exists $G \in \mathfrak{G}$ such that the policy function of $G$ exhibits a 3-period cycle.*

This remarkable result is typically characterized as an "exact discount factor restriction" in optimal dynamics and understood and abbreviated as a necessary and sufficient condition for complicated dynamics for a general but precisely-specified class of models. It is a remarkable result in that by virtue of the fact that the technological specification of the considered class of models eschews explicit functional forms and is delineated only by assumptions expressed in a qualitative postulational form, a specific

☆ It is with strong feelings of both sadness and satisfaction that the first two authors submit this manuscript for publication: sadness because the gifts and talents of Tapan Mitra are no longer with us, satisfaction because it brings to completion a project that began in 2010 but whose principal result was derived in active correspondence with him during the last five months of his life, and regarding which he retained an enthusiasm till the end. In the sequel, we reserve a section in the Appendix on an optimization problem to make salient his precise breakthrough. The responsibility for any blemishes in this is however solely ours. This final version has benefited from the erudition and careful reading of an anonymous referee. Liuchun Deng acknowledges the support of the Start-up Grant from Yale-NUS College, Singapore.

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1 See *Connoisseur of Chaos* (1997), p.194 in F. Kermode and J. Richardson (eds.) *Wallace Stevens: Collected Poetry and Prose.* New York: The Library of America. Also see *Chaos in Motion and Not in Motion*, p. 311. In this and the subsequent quotation below, we disregard line-breaks and the consequent punctuation.

2 See *The Misfit* (2017), p.132 in R. M. West (ed.) *The Complete Poems of A. R. Ammons*, Volume 1. New York: W. W. Norton and Company. Also see *Chaos Staggered Up the Hill*, p.500; and Tobin (1999).

3 See Mitra (1996) and Nishimura and Yano (1996). Since we are emphasizing "simultaneity", the reader should note that careful inspection of the sets of assumption of the two models, as in the footnote below, yields that the results are not identical: however, for the purposes of this paper, they are similar enough.

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number for the discount factor can nevertheless be furnished — Mitra refers to this number as a "universal" constant. Furthermore, to the extent that complicated dynamics are represented by 3-period cycles, it is testimony to the importance of the Li–Yorke theorem (Li and Yorke, 1975) of chaotic dynamics on the unit interval. The result itself followed an earlier opening by Sorger (1994) and has led to a rich trajectory of work (Sorger, 2018) and by now has gone beyond the investigation of 3-period cycles into a body of work that Mitra and Sorger (1999) and their followers refer to as "rationalizability conditions for dynamic optimization problems".

The question is how general is the general class of models that this literature addresses. To be sure, in keeping with the one-dimensional rubric of chaotic dynamics, the models are already limited to a single capital stock, but disregarding this, their assumptions on the technological specification do not go very much beyond the requirements of closedness, convexity, free-disposal, inaction and the impossibility of an output without an input; and those on the planner's preferences beyond continuity, monotonicity and a version of strict concavity.[4] The point is that this admittedly general setting rules out the two-sector RSS model.[5] Even though not a special case, the question remains whether the tools and techniques pioneered by MNY carry over in a routine way to deliver an analogous result for the RSS case? And if so, is it at all obvious that the same universal constant would then be obtained? And indeed whether the inquiry itself could be broadened to a model with explicit functional forms to take the question beyond the discount factor to other explicit parameters of the model: to labor productivity and to the depreciation rate, the other two parameters of the RSS specification, for the example considered here?

In this paper we address all of these questions. However, a basic point regarding motivation needs clarification at the outset: our investigation is not simply to domesticate an erratic special case falling outside the general scope of the theory, but rather an elaboration and underscoring of May's 1976 dictum of simple explicitly-specified models as being important litmus tests of general results in dynamical systems and economic dynamics.[6] And leaving aside any question as to the nature of the argumentation, what we find is indeed surprising. First, there is a universal constant for the RSS model but one different from the one discovered earlier for the "general" case: rather than $\bar{\rho}$, it is $\hat{\rho} = ((\sqrt{3}-1)/2) < \bar{\rho}$. This constitutes our first theorem. Second, and perhaps even more importantly, the specificity of the RSS conception allows one to devalorize the discount-factor; in short to emphasize labor productivity and capital durability in a measure equal to the rate of impatience. However, rather than a necessary and sufficient condition (in the sense of Theorem MNY above) for each of these parameters, we identify a necessary and sufficient *region* of the two parameters. This constitutes our second theorem: its formulation involves subtle differences from our first result, and we address them in some detail in the sequel.

In their investigation of the RSS model originally initiated in 2005, Khan–Mitra provided in 2010 an "explicit solution of the optimal policy function (henceforth OPF) when the discount factor is less than the labor-output ratio", and used that solution to follow up their earlier result on the existence of optimal topological chaos for the model in a way that only a few qualitative observations of the OPF needed to be utilized for this purpose; see Khan and Mitra (2005a,b). Using the more detailed information on the OPF subsequently available, they could establish optimal chaotic dynamics for a non-negligible parametric ranges of the model.[7] In that study, they also touched on the questions being posed here but in a diffused non-conclusive way. They considered both 3-period cycles and turbulence, as formalized by Block–Coppel–Misurewicz using Smale's work as a point of departure,[8] but what is of consequence and relevance to this paper is that in the earlier study, they could not obtain an *exact* discount-factor restriction for 3-period cycles. However, they did obtain an exact restriction for the labor-output ratio, one of the other technological parameters of the model. Our Theorem 1 is the result that eluded them, and our Theorem 2 is a far-reaching generalization of their second result. After presenting what we see to be definitive results, we further discuss their relation to the earlier exploratory ones in Section 3 so that the marginal contribution of this paper relative to theirs can be fully gauged. We also discuss the novelty of the argumentation provided here as compared to the earlier proofs. After a brief recapitulation of the RSS model in Section 2, this is done in Section 5 on the proofs and the ancillary results that they rely on. We also include, for the convenience of the reader, several known results on the properties of the optimal policy from Khan and Mitra (2007, 2012, 2020) in Section 5. Section 4 is a two-remark concluding section spelling out open questions.

## 2. The model and antecedent results

We consider the two-sector discrete-time RSS model of optimal growth with discounting. There are two sectors in the economy. In the consumption good sector, it requires a unit of labor and a unit of investment good to produce a unit of the consumption good. In the investment good sector, only labor is required to produce the investment good. In particular, it requires $a > 0$ units of labor to produce a unit of investment good. Capital depreciates at the rate of $d \in (0, 1)$. A constant amount of labor, normalized to unity, is available in each time period $t$. Denote the capital stock in the current period (today) by $x$ and that in the next period (tomorrow) by $x'$. The *transition possibility set* then takes the specific form

$$\Omega = \{(x, x') \in \mathbb{R}^2_+ : x' - (1-d)x \geq 0 \text{ and } a(x' - (1-d)x) \leq 1\}.$$

Associated with $\Omega$ is the transition correspondence, $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, given by $\Gamma(x) = \{x' \in \mathbb{R}_+ : (x, x') \in \Omega\}$. For any $(x, x') \in \Omega$. Denote by $y$ the amount of capital stock available for the production of the consumption good. We have

$$\Lambda : \Omega \longrightarrow \mathbb{R}_+ \text{ with } \Lambda(x, x')$$
$$= \{y \in \mathbb{R}_+ : 0 \leq y \leq x \text{ and } y \leq 1 - a(x' - (1-d)x)\}.$$

Welfare is derived only from the consumption good and is represented by a linear function, normalized so that $y$ units of the consumption good yield a welfare level $y$. A *reduced form utility function* is given by

$$u : \Omega \to \mathbb{R}_+ \text{ with } u(x, x') = \max\{y \in \Lambda(x, x')\}$$

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4 The key difference between Mitra (1996) and Nishimura and Yano (1996) lies in their assumptions on the reduced form utility function. In Mitra (1996), the utility function is required to be strictly concave with a relatively weak monotonicity assumption, while in Nishimura and Yano (1996), the utility function is required to be strictly increasing in its first argument, strictly decreasing in its second argument, concave but not necessarily strictly concave.

5 A1(ii), A4 and A5 in Mitra (1996) are violated; in particular, the reduced-form utility function is not strictly concave (A5 not satisfied). Note that A4 is in fact not necessary for his main results in Mitra (1996). As regards A2 in Nishimura and Yano (1996), the strict monotonicity assumption does hold for the RSS model. The claim here is that their theorems as stated do not apply to the RSS model considered here; this is not to say that their methods of proof could not be applied to the proof of Theorem 1 in some ingenious manner.

6 See May (1976, 2001) and May and Oster (1976) for stressing the importance of simple deterministic discrete-time models to understand complicated dynamics.

7 The paper was published only in 2020; see Khan and Mitra (2020).

8 See Smale (1967), Misiurewicz and Block and Coppel (1992).

The intertemporal preference is represented by the present value of the stream of welfare levels with a discount factor $\rho \in (0, 1)$. In the sequel production plans in the set $\Omega$ and in the domain of $\Lambda$ will also be generically denoted as $(x(t), x(t+1))$ when we want to emphasize the temporal aspect.

A two-sector RSS model consists of a triplet $(a, d, \rho)$. A *program* $\{x(t), y(t)\}$ from $x_0$ is a sequence $\{x(t), y(t)\}$ such that $x(0) = x_0$, and for all $t \in \mathbb{N}$, $(x(t), x(t+1)) \in \Omega$ and $y(t) = \max \Lambda(x(t), x(t+1))$. A program $\{x(t), y(t)\}$ is *stationary* if for all $t \in \mathbb{N}$, $(x(t), y(t)) = (x(t+1), y(t+1))$. A program $\{\bar{x}(t), \bar{y}(t)\}$ from $x_0$ is called *optimal* if

$$\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) \leq \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))$$

for every program $\{x(t), y(t)\}$ from $x_0$. The parameter $\xi \equiv (1/a) - (1-d)$ features prominently in the subsequent analysis. It represents the marginal rate of transformation of capital today into that of tomorrow under full utilization of resources. In what follows, and without further mention, we always assume that the parameters $(a, d)$ of the RSS model are such that[9]

$$\xi > 1.$$

Using the standard methods of dynamic programming, one can establish that there exists an optimal program from every $x \in X \equiv [0, \infty)$, and then use it to define a *value function*, $V : X \to \mathbb{R}$ by

$$V(x) = \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1)),$$

where $\{\bar{x}(t), \bar{y}(t)\}$ is an optimal program from $x$. It can be shown that $V$ is concave, increasing and continuous on $X$; see Khan and Mitra (2007) for details. Furthermore, the Bellman functional equation holds: this is to say that for each $x \in X$, the following equation

$$V(x) = \max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\}$$

holds. For $\rho \in (0, 1)$, the value function $V$ is the unique continuous function on $Z \equiv [0, (1/ad)]$ which satisfies the Bellman equation. For each $x \in X$, we define the *optimal policy correspondence (OPC)*

$$h(x) = \arg\max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\}.$$

Then, a program $\{x(t), y(t)\}$ from $x \in X$ is an optimal program from $x$ if and only if $x(t+1) \in h(x(t))$ for $t \geq 0$. When the *OPC* is a function, we refer to it as the *optimal policy function (OPF)*.

A *modified golden rule* is a pair $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$ such that $(\hat{x}, \hat{x}) \in \Omega$ and

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x) \text{ for all } (x, x') \in \Omega.$$

Given a modified golden-rule $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$, we know that $\hat{x}$ is a stationary optimal stock. The following proposition establishes the existence of and the closed-form solution to the modified golden rule.

**Proposition 1** (Khan and Mitra, 2006a, 2007)**.** *Define* $(\hat{x}, \hat{p}) = (1/(1+ad),\ 1/(1+\rho\xi))$. *Then* $(\hat{x}, \hat{x}) \in \Omega$, *where* $\hat{x}$ *is independent of* $\rho$, *and satisfies the modified golden rule.*

We summarize below the basic properties of the *OPC*. Those properties will be used in the proofs of our main results. To this end, we describe three regions of the state space:

$$A = [0, \hat{x}],\ \ B = (\hat{x}, k),\ \ C = [k, \infty)$$

where $k = \hat{x}/(1-d)$. We further subdivide the region $B$ into two regions as follows:

$$D = (\hat{x}, 1),\ \ E = [1, k)$$

and define a correspondence, $G : X \to X$, by:

$$G(x) = \begin{cases} \{(1/a) - \xi x\} & \text{for } x \in A \\ [(1/a) - \xi x, \hat{x}] & \text{for } x \in D \\ [(1-d)x, \hat{x}] & \text{for } x \in E \\ \{(1-d)x\} & \text{for } x \in C \end{cases}$$

**Proposition 2** (Khan and Mitra, 2007)**.** *The OPC, h, satisfies* $h(x) \subset G(x)$ *for any* $x \in X$.

The graph of the correspondence $G$ is given by the red-shaded are in Fig. 1.[10] This result suggests that the only part of the *OPC* for which we do *not* have an explicit solution is for the middle region of stocks, given by $B = (\hat{x}, k) = D \cup E$. Moreover, if the discount factor is smaller than the labor-output ratio in the investment good sector, i.e., $\rho < a$, with $a < 1$ guaranteed by the standing hypothesis $\xi > 1$, we can fully characterize the *OPC*. It is now a function, and is referred to as the *check-map*: it is defined as follows

$$H(x) = \begin{cases} (1/a) - \xi x & \text{for } x \in [0, 1] \\ (1-d)x & \text{for } x \in (1, \infty) \end{cases}$$

**Proposition 3** (Khan and Mitra, 2012, 2020)**.** *If* $\rho < a$, *then the OPC, h, is given by H.*

## 3. Exact parametric restrictions for 3-period cycles

In the MNY Theorem, the reduced-form utility function $u$ is required to be either (i) strictly concave on its domain or (ii) strictly increasing in its first argument and strictly decreasing in its second argument.[11] As explained in Khan and Mitra (2020), the reduced-form utility function in the RSS model is not strictly concave in either the first or the second argument. For the reader's convenience, we reproduce their formal reasoning here. Consider $x, \bar{x}$ with $1 < x < \bar{x} < k$, and $(x', \bar{x}') = (1-d)(x, \bar{x})$. Pick $(\tilde{x}, \tilde{x}')$ such that $\tilde{x} = \lambda x + (1-\lambda)\bar{x}$ and $\tilde{x}' = \lambda x' + (1-\lambda)\bar{x}'$ with any $\lambda \in (0, 1)$. We have $(x, x') \in \Omega$, and $(\bar{x}, \bar{x}') \in \Omega$, and $(\tilde{x}, \tilde{x}') \in \Omega$. However, $u(x, x') = u(\bar{x}, \bar{x}') = u(\tilde{x}, \tilde{x}') = 1$. Moreover, for $(x, x') \in \Omega$ such that $x < 1 - a(x' - (1-d)x)$, $u(x, x') = x$, so $u$ is not strictly decreasing in its second argument. In short, the two-sector RSS model is not an element in the general class of optimal growth models considered by the MNY Theorem.

### 3.1. An exact discount factor restriction

We can now present the first principal result of the paper.

**Theorem 1.** *Let* $\hat{\rho} \equiv \frac{\sqrt{3}-1}{2}$. *(i) If* $\rho < \hat{\rho}$, *then there exist* $a \in (0, 1)$ *and* $d \in (0, 1)$ *such that the RSS model with parameters* $(a, d, \rho)$ *has an OPF which generates a 3-period cycle. (ii) If the RSS model with parameters* $(a, d, \rho)$ *has an OPF which generates a 3-period cycle, then* $\rho < \hat{\rho}$.

9 For more details, the reader is referred to Khan and Mitra (2006b, 2007). Other than Fig. 1, we abstain from any diagrammatic rendering of the model: any reader inclined towards geometry needing additional details regarding the figure can see Khan and Mitra (2006b, 2013) and their antecedent references.

10 For details on this, and the check-map *VMD* specified in Proposition 3, see Khan and Mitra (2006a, 2013).

11 For (i), see Assumption A5 in Mitra (1996); for (ii), see Assumption A2 in Nishimura and Yano (1996).

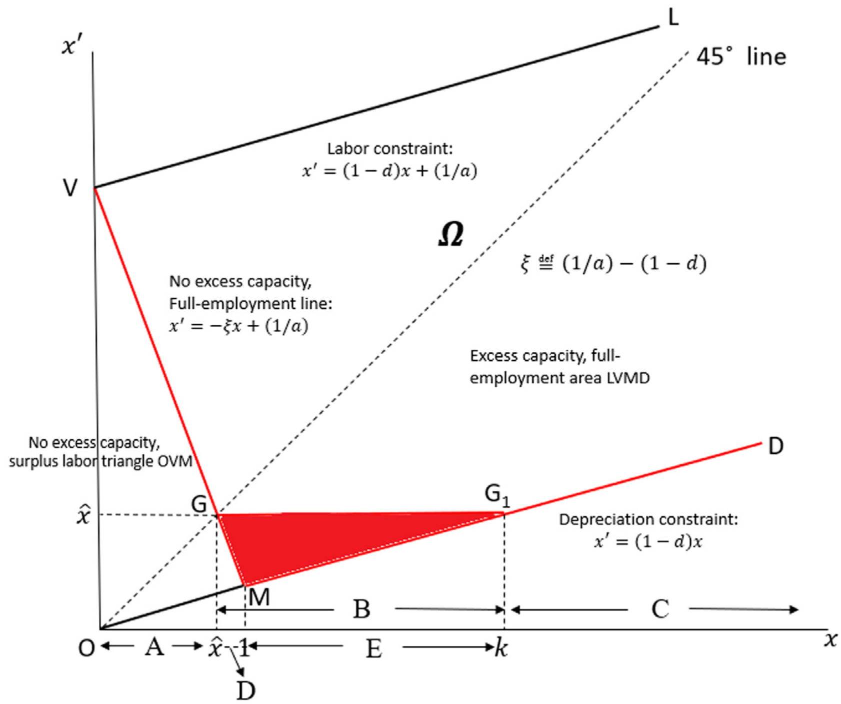


**Fig. 1.** The optimal policy correspondence.

We invite the reader to compare Theorem 1 with Theorem MNY that began this exposition. In particular, it is worth emphasizing that the least upper bound on the discount factor we have identified $\hat{\rho}$ is strictly smaller than the $\bar{\rho}$ that they identified for a general class of growth models. Our theorem shows how the exact factor restriction can be further sharpened when specific structures of a model are imposed.

We turn to a sketch of the proof. The proof of part (i) consists of two essential steps. First, for a given discount factor $\rho < \hat{\rho}$, we pick a depreciation rate $d$ sufficiently close to zero and a corresponding technological parameter $a$ such that the resulting check map has a 3-period cycle starting from the bottom (1, $H(1)$, and $H^2(1)$). Next, we prove that with $d$ sufficiently close to zero, the check map is indeed the optimal policy function for the two-sector RSS model $(a, d, \rho)$.

The proof of part (ii) is more involved. First, for any 3-period cycle, we can show that the three periodic points are in the three regions $A$, $B$, and $C$, respectively; see Fig. 1. Second, exploiting the 3-period cyclical structure and the slope of the value function, we obtain the necessary conditions concerning $\rho$, $\xi$, and $d$. Last, the upper bound of the discount factor turns out to be solution to an optimization problem which arises from the necessary conditions for 3-period cycles.

To highlight the difference between an exact and a non-exact discount factor restriction, we reproduce Proposition 9 in Khan and Mitra (2020).

**Proposition 4** (*Khan and Mitra, 2020*)**.** *(i) If $0 < \rho < (1/3)$, then there exist $a \in (0, 1)$ and $d \in (0, 1)$ such that the RSS model with parameters $(a, d, \rho)$ has an OPF which generates a 3-period cycle. (ii) If the RSS model with parameters $(a, d, \rho)$ has an OPF which generates a 3-period cycle, then $\rho < (1/2)$.*

Since $1/3 < \hat{\rho} < 1/2$, our discount factor restriction as in Theorem 1 is "exact" as opposed to that in the above result: the theorem imposes the least upper bound on the discount factor for the existence of 3-period cycles: for any discount factor below $\hat{\rho}$, we can construct an RSS model whose *OPF* generates a 3-period cycle, and for any RSS model whose *OPF* generates a 3-period cycle, the discount factor has to be below $\hat{\rho}$.

### *3.2. An exact region of parameters concerning labor productivity and depreciation*

We can now present the second principal result of the paper. It establishes an exact region of two parameters concerning labor productivity and depreciation for the existence of 3-period cycles. To the best of the authors' knowledge such an "exact" region has not been presented before in optimal economic dynamics.[12]

**Theorem 2.** *(i) If $(\xi - 1)(1-d) \geq 1$, then there exists $\rho^*$ such that the RSS model with parameters $(a, d, \rho^*)$ has an OPF which generates a 3-period cycle. (ii) If the RSS model with parameters $(a, d, \rho^*)$ has an OPF which generates a 3-period cycle, then $(\xi - 1)(1-d) \geq 1$.*

Part (i) of the theorem immediately follows from Proposition 5 in Khan and Mitra (2020) by choosing $\rho < a$. Part (ii) of the theorem follows from Lemma 5. Theorem 2 concerns a region of the two parameters for 3-period cycles, an exercise that cannot be conducted for a general class of growth models in which the discount factor is the only numerical parameter. It is also worth pointing out that we characterize the exact region through the parameters $\xi$ and $d$ : as in Khan and Mitra (2013, Figure 8), an alternative characterization is given by

$$(\xi - 1)(1-d) \geq 1 \Longleftrightarrow (1/a) \geq 1 + (1-d) + (1-d)^{-1}.$$

We now show how Theorem 2 straightforwardly yields Proposition 6 and 7 in Khan and Mitra (2020).

**Corollary 1.** *(i) Let $0 < a < (1/3)$. Then, there exist $\rho \in (0, 1)$ and $d \in (0, 1)$ such that the RSS model with parameters $(a, d, \rho)$ has an OPF which generates a 3-period cycle. (ii) If the RSS model with parameters $(a, d, \rho)$ has an OPF which generates a 3-period cycle, then $a < (1/3)$.*

[12] Proposition 8 in Khan and Mitra (2020) provides a similar result for the existence of the second iterate of the optimal policy function being turbulent but the underlying idea of an "exact" region of restrictions seems to have been missed: in any case it is neither elaborated nor given any salience.

This corollary is a counterpart of the exact discount factor restriction for the technological parameter, $a$. The cutoff $(1/3)$ can be easily derived from the region for two parameters, $(\xi-1)(1-d) \geq 1$.[13] It should, however, be pointed out that the region imposes no restriction on the depreciation rate, $d$, per se. In fact, any $0<d<1$ can be compatible with optimal 3-period cycles.

**Remark 1.** We state our Theorems 1 and 2 for 3-period cycles generated by an OPF. Since our proofs do not rely on $h$ being a function, if an OPC, which is not an OPF, generates a 3-period cycle, then we still must have $\rho < \hat{\rho}$ and $(\xi-1)(1-d) \geq 1$.[14]

## 4. Concluding remarks

In terms of a summary statement that connects to chaotic dynamics and to the epigraphs, this paper reports two results that pertain to the existence of 3-period cycles. It obtains an exact discount factor restriction for the two-sector RSS model different from that obtained in the MNY Theorem for a general class of models. It is not a "universal constant" but firmly tethered to a model of optimal growth, example if one prefers, that is on its way to attaining a canonical status. Furthermore, the exact discount factor restriction can be extended to an exact *region* delineated by labor productivity and depreciation, two additional parameters of the RSS model that are uniquely its own. In terms of the general class of models that MNY work with, as well as the ones Mitra–Sorger do for the rationalizability-of-particular-maps variant, the discount factor is the *only* numerical parameter. Our result then goes beyond the singularity of the discount factor in the general setting and brings out how exact parametric restrictions can be concretized more generally in a specific instance of theory of intertemporal resource allocation.

We conclude with two further remarks. There has been a recent revival of interest in the two-sector Robinson–Shinkai–Leontief (RSL) model of optimal growth, which nests the RSS model in this paper as a special case (Deng et al., 2019; Deng and Fujio, 2020). In contrast to the RSS model, if capital is also required to produce the investment good, then the monotonicity conditions of the reduced-form utility in Nishimura and Yano (1996) are met and thus their discount factor restriction applies. It nevertheless remains an open question how to characterize the exact discount factor restriction in the RSL model. Second, as far as complicated dynamics is concerned, one can go considerably beyond the Li–Yorke condition. In the context of the RSS model, as already stated above in Footnote 12, Khan and Mitra (2020) have provided (non-exact) discount factor restriction for the second iterate of the optimal policy function being turbulent. Their result opens a new door for inquiry: it leads one to ask for parametric restrictions pertaining to odd-period cycles or indeed, other alternative representations of complicated dynamics. We leave these questions for future investigation.

## 5. Proofs of the results

In this section, we turn to the proofs of Theorems 1 and 2. We begin with four preliminary lemmata. The first is a collection of several known results on the properties of the optimal policy of the RSS model taken from Khan and Mitra (2007, 2012, 2020) and reproduced here for the reader's convenience. The second is one of the three canonical cases of the 2-sector RSS model and whose geometry is considered in Khan and Mitra (2013, Figure 8).

**Lemma 1.** *The optimal policy correspondence $h$ has the following properties:*

*(1) If $\hat{x} \in h(x)$ for some $x \in B$, then $\hat{x} \in h(x)$ for every $x \in B$.*
*(2) If $\hat{x} \in h(x)$ for some $x \in (\hat{x}, k)$, then $V'_-(\hat{x}) \geq (a/\rho)$.*
*(3) If $\rho\xi < 1$, then for every $x \in (\hat{x}, k)$, $\hat{x} \notin h(x)$.*
*(4) If $\rho\xi > 1$, then for every $x \in (\hat{x}, k)$, $h(x) = \{\hat{x}\}$.*
*(5) If $V'_+(H(x)) \leq a/\rho$ for some $x \in B$, then $H(x) \in h(x)$.*
*(6) If $V'_-(\hat{x}) < (a/\rho)$ and $V'_+(1-d) > (a/\rho)$, then there exist $m$ and $n$ satisfying $\hat{x} < m < 1 < n < k$ such that the following conditions hold:*
*(a1) $V'_+(H(x)) \leq (a/\rho)$ for all $x \in (\hat{x}, m] \cup [n, k)$;*
*(a2) $V'_+(H(x)) > (a/\rho)$ for all $x \in (m, n)$;*
*(b1) $H(m) = H(n) \in h(x)$ for all $x \in [m, n]$;*
*(b2) $H(x) \in h(x)$ for all $x \in (\hat{x}, m] \cup [n, k)$;*
*(c) $V$ is linear on $[m, n]$ with $V'(x) = a(1-d)$ for all $x \in (m, n)$.*
*(7) If $h(\tilde{x}) = H(\tilde{x})$ for some $\tilde{x} \in (\hat{x}, 1]$, then $h(x) = H(x)$ for all $x \in [\hat{x}, \tilde{x}]$.*

**Proof.** (1), (2), (3), and (4) are Lemmata 4 and 6 Propositions 3 and 4 and Remark (i) in Khan and Mitra (2007). (5) and (6) are Lemma 1 and Proposition 6 in Khan and Mitra (2012). (7) is Lemma 6 in Khan and Mitra (2020). ■

**Lemma 2.** *If $(\xi-1)(1-d) = 1$, then the check map $H$ has a 3-period cycle: $1$, $(1-d)$, $(1/a) - \xi(1-d)$.*

**Proof.** Since $(\xi-1)(1-d) = 1$, we have:

$$\left.\begin{array}{l} H(1) = (1-d) \\ H^2(1) = H(1-d) = (1/a) - \xi(1-d) = 1/(1-d) > 1 \\ H^3(1) = H(H^2(1)) = (1-d)[(1/a) - \xi(1-d)] = 1 \end{array}\right\}$$

where $H$ is the check map. The details of the last line of the equation above are as follows:

$$\begin{aligned} H^3(1) &= (1-d)[(1/a) - \xi(1-d)] \\ &= (1-d)(1/a) - \xi(1-d)^2 \\ &= (1-d)(1/a) - (1-d)[1+(1-d)] \\ &= (1-d)[(1/a) - (1-d) - 1] \\ &= (1-d)(\xi-1) = 1 \end{aligned}$$

where the third and the fifth equality follow from $(\xi-1)(1-d) = 1$.

Therefore, $H$ has a 3-period cycle: $1$, $(1-d)$, $(1/a) - \xi(1-d)$. ■

Next, we turn to a routine arithmetical consequence.

**Lemma 3.** *If $\rho^3\xi a(1-d)^2 - \rho + a < 0$, then $\xi(1-d)\rho^2 + \xi\rho - 1 > 0$*

**Proof.** We have

$$\begin{aligned} g(\rho) &\equiv \rho^3\xi a(1-d)^2 - \rho + a \\ &= \rho^3\xi a(1-d)^2 - \xi a\rho + \xi a\rho - \rho + a \\ &= \rho a\xi[\rho^2(1-d)^2 - 1] - \rho(1 - a\xi) + a \\ &= \rho a\xi[\rho^2(1-d)^2 - 1] - \rho a(1-d) + a \\ &= \rho a\xi[\rho^2(1-d)^2 - 1] - a[\rho(1-d) - 1] \\ &= [\rho(1-d) - 1]\{\rho a\xi[\rho(1-d) + 1] - a\}. \end{aligned}$$

Since $\rho(1-d) - 1 < 0$, and we know $g(\rho) < 0$, we have

$$\rho a\xi[\rho(1-d) + 1] - a > 0 \Longrightarrow \xi(1-d)\rho^2 + \xi\rho - 1 > 0,$$

to complete the verification. ■

[13] A simple proof is offered in Section 5.

[14] We thank our anonymous referee for raising this question. For a more detailed elaboration, see Remark 2 in Section 5.

Next, we turn to a result on a lower bound on the left-hand derivative of the value function when the capital stock lies in the intervals designated by region $B$ in Fig. 1.

**Lemma 4.** *$V'_-(\hat{x}) \geq (a/\rho)$ if and only if $\hat{x} \in h(x)$ for some $x \in (\hat{x}, k)$*

**Proof.** The "if" part follows (2) of Lemma 1. So we prove the "only if" part. Suppose $V'_-(\hat{x}) \geq (a/\rho)$, and let $x \in (\hat{x}, k)$ be given. Then we have $\hat{x} = (1-d)[\hat{x}/(1-d)] > (1-d)x$, and $a[\hat{x}-(1-d)x] < a[\hat{x}-(1-d)\hat{x}] = ad\hat{x} = ad/(1+ad) < 1$. Further, we have:

$$\begin{aligned}1 - a[\hat{x}-(1-d)x] &= 1 - a[\hat{x}-(1-d)\hat{x}] + a(1-d)(x-\hat{x})\\ &= \hat{x} + a(1-d)(x-\hat{x})\\ &< \hat{x} + (x-\hat{x})\\ &= x\end{aligned}$$

where the inequality follows from $\xi > 1$. Thus, there is $\varepsilon > 0$, such that for all $z \in I \equiv (\hat{x}-\varepsilon, \hat{x}+\varepsilon)$, we have $(x,z) \in \Omega$ and $\{1 - a[z-(1-d)x]\} < x$, so that:

$$u(x,z) = 1 - a[z-(1-d)x]$$

Define $F(x) = \{z : (x,z) \in \Omega\}$, and for $z \in F(x)$, define:

$$W(z) = u(x,z) + \rho V(z)$$

Clearly, $W$ is concave on its domain. For $z \in I$, we have:

$$W(z) = 1 - az + a(1-d)x + \rho V(z)$$

Since $\hat{x} \in I$, we obtain:

$$W'_+(\hat{x}) = -a + \rho V'_+(\hat{x}) \leq 0 \tag{1}$$

the inequality in (1) following from $V'_+(\hat{x})\hat{p} < a/\rho$ (see Eq. (19) in Khan and Mitra, 2007). And, we can obtain:

$$W'_-(\hat{x}) = -a + \rho V'_-(\hat{x}) \geq 0 \tag{2}$$

the inequality in (2) following from $V'_-(\hat{x}) \geq (a/\rho)$.

Now for all $z \in F(x)$ with $z > \hat{x}$, we have:

$$W(z) - W(\hat{x}) \leq W'_+(\hat{x})(z-\hat{x}) \leq 0$$

by using concavity of $W$ and (1). Similarly, for $z \in F(x)$ with $z < \hat{x}$, we have:

$$W(z) - W(\hat{x}) \leq W'_-(\hat{x})(z-\hat{x}) \leq 0$$

by using the concavity of $W$ and (2). Thus, we have $W(z) \leq W(\hat{x})$ for all $z \in F(x)$. This means:

$$\max_{(x,z)\in\Omega} [u(x,z) + \rho V(z)] = u(x,\hat{x}) + \rho V(\hat{x})$$

Since the left-hand side expression in the equation above is $V(x)$ by the optimality principle, we have:

$$V(x) = u(x,\hat{x}) + \rho V(\hat{x})$$

This means that $\hat{x} \in h(x)$. ■

We can now finally turn to the proof of Theorem 1. We begin with the proof for part (i) of the theorem.

**Proof of Theorem 1 Part (i).** We break up the proof into two separate steps.

**Step 1.** For any $d \in (0,1)$, we can define:

$$\xi(d) = 1 + \frac{1}{(1-d)} > 1, \quad a(d) = \frac{1}{\xi(d)+(1-d)},$$

$$\xi(d) = \frac{1}{a(d)} - (1-d),$$

so that $a(d) \in (0,1)$ and $(\xi(d)-1)(1-d) = 1$. Consider for any given $d \in (0,1)$, the function:

$$f(r) = \xi(d)(1-d)r^2 + \xi(d)r - 1 \quad \text{for all } r \in \mathbb{R} \tag{3}$$

Clearly $f(0) = -1$ and $f(r) \to \infty$ as $r \to \infty$. So, there is some $\hat{r}(d) > 0$ such that $f(\hat{r}(d)) = 0$. Since $f$ is increasing for $r \geq 0$, this solution is unique. It is given by:

$$\hat{r}(d) = \frac{-\xi(d) + \sqrt{\xi(d)^2 + 4\xi(d)(1-d)}}{2\xi(d)(1-d)}$$

Note that as $d \to 0$, we have $\xi(d) \to 2$. Using this in the equation above, we infer that as $d \to 0$, we must have $\hat{r}(d)$ converge to $\hat{r} \equiv \frac{-2+\sqrt{12}}{4} = \hat{\rho} = \frac{\sqrt{3}-1}{2}$.

Since $\rho < \hat{\rho} < (1/2)$, we can choose $d' \in (0,1)$, with $d'$ sufficiently close to zero, such that with $\xi(d')$ and $a(d')$ defined as above, we will have $\hat{\rho}\xi(d') < 1$, and $\rho < \hat{r}(d')$. We fix this $d'$ in what follows and (to simplify notation) call it $d$. The corresponding $\xi(d)$ is called $\xi$, and the corresponding $a(d)$ is called $a$. To summarize, we now have $(a,d)$, such that $\xi = (1/a) - (1-d)$, $\xi\hat{\rho} < 1$, $(\xi-1)(1-d) = 1$, and $\rho < \hat{r}(d)$. According to Lemma 2, $(\xi-1)(1-d) = 1$ implies that the check map $H$ has a 3-period cycle: $1, (1-d), (1/a) - \xi(1-d)$. It remains to show that under condition $\rho < \hat{r}(d)$, the optimal policy function for the RSS model $(a,d,\rho)$ is given by $H$, which generates a period three cycle.

**Step 2.** Suppose the check map is not optimal. We assert that there exists a flat-bottom map (with a narrower bottom than the pan map $VGG_1D$ as illustrated in Fig. 1) which is optimal.[15] Since $\rho\xi < \hat{\rho}\xi < 1$, we have $\hat{x} \notin h(x)$ for any $x \in (\hat{x}, k)$ by (3) of Lemma 1 which further implies $V'_-(H(k)) = V'_-(\hat{x}) < a/\rho$ by Lemma 4. Since the check map is not optimal, we also have $V'_+(H(1)) > a/\rho$ by (5) of Lemma 1. Then by (6) of Lemma 1, there exist $m, n$ satisfying $\hat{x} < m < 1 < n < k$, such that $H(x) \in h(x)$ for all $x \in (\hat{x}, m] \cup [n, k)$, $H(m) = H(n) \in h(x)$ for all $x \in [m,n]$, and $V$ is linear on $[m,n]$ with $V'(x) = a(1-d)$ for all $x \in (m,n)$.

We now consider an optimal program from $s = (1-d) = H(1)$. Clearly, $h(s) = (1/a) - \xi s = H^2(1)$, so that the proof of Lemma 2 implies that $h(s) \in (k, \infty)$. Thus, $h^2(s) = (1-d)h(s) = H^3(1) = 1$, so that $h^2(s) \in (m,n)$.

Similarly, by considering an optimal program from $s' > s$, and sufficiently close to $s$, one has $h(s') = (1/a) - \xi s' \in (k, \infty)$, and $h^2(s') = (1-d)h(s') \in (m,n)$. Thus, we have:

$$V(s) = s + \rho + \rho^2 V(1) \quad \text{and}$$

$$V(s') = s' + \rho + \rho^2 V(1 - \xi(1-d)(s'-s)),$$

which leads to

$$\frac{V(s')-V(s)}{(s'-s)} = 1 - \rho^2\xi(1-d)\frac{[V(1) - V(1-\xi(1-d)(s'-s))]}{\xi(1-d)(s'-s)}$$

Letting $s' \to s$ in the equation above, we get:

$$V'_+(s) = 1 - \rho^2\xi(1-d)V'_-(1)$$

Since we know $V'(x) = a(1-d)$ for all $x \in (m,n)$, we must have $V'_-(1) = a(1-d)$, and using this in the equation above, we get:

$$V'_+(s) = 1 - \rho^2\xi a(1-d)^2$$

Since we are supposing that the check map is not optimal, $V'_+(s) = V'_+(1-d) = V'_+(H(1)) > (a/\rho)$ by (5) of Lemma 1. Using this in the equation above, we get:

$$a/\rho < 1 - \rho^2\xi a(1-d)^2,$$
$$\text{or equivalently,}\quad \rho^3\xi a(1-d)^2 - \rho + a < 0,$$

[15] This closely follows from the proof of Theorem 3 in Khan–Mitra (Khan and Mitra, 2012).

which by Lemma 3 implies

$$\xi(1-d)\rho^2 + \xi\rho - 1 > 0.$$

Using the notation introduced in (3), $f(\rho) > 0$. This implies $\rho > \hat{r}(d)$, which contradicts $\rho < \hat{r}(d)$ [recalling that we agreed to call $d'$ in $\rho < \hat{r}(d)$ as $d$]. Thus, the check map is optimal, and the 3-period cycle $(1, (1-d), (1/a) - \xi(1-d))$ is an optimal 3-period cycle.

This completes the proof. ■

The proof of Part (ii) of Theorem 1 requires a necessary condition for 3-Period Cycles. The interested reader might want to track the argumentation of the proof through Fig. 1.

**Lemma 5.** *If the optimal policy function generates a 3-period cycle, then*

$$(\xi - 1)(1-d) \geq 1$$

**Proof.** Denote the optimal policy function by $h$, and the 3-period cycle stocks by $\alpha, \beta, \gamma$. Without loss of generality we may suppose that $\alpha < \beta < \gamma$. Note that none of the three points can be equal to $\hat{x}$, since $\hat{x}$ is a fixed point of $h$. Similarly, none of the points can be equal to $k$, since $h(k) = \hat{x}$. In view of this, it will help in our exposition if we define the sets $A' = [0, \hat{x})$ and $C' = (k, \infty)$.

There are then two possibilities to consider: (i) $\beta = h(\alpha)$; (ii) $\gamma = h(\alpha)$. In case (i), we must have $\alpha \in A'$, since $\beta > \alpha$. Consequently, $\beta = (1/a) - \xi\alpha$, and $\gamma \neq h(\alpha)$. Thus, we must have $\gamma = h(\beta)$, and since $\gamma > \beta$, we must have $\beta \in A'$. But, since $\beta = (1/a) - \xi\alpha$ with $\alpha \in A'$, we must have $\beta \in B \cup C'$, a contradiction. Thus, case (i) cannot occur.

Thus case (ii) must occur. In this case, since $\gamma > \alpha$, we must have $\alpha \in A'$. Consequently, $\gamma = (1/a) - \xi\alpha$, and $\beta \neq h(\alpha)$. Thus, we must have $\beta = h(\gamma)$; it also follows that we must have $\alpha = h(\beta)$. Since $\beta < \gamma$, we must have $\gamma \in B \cup C'$; similarly, since $\alpha < \beta$, we must have $\beta \in B \cup C'$.

We claim now that $\gamma \in C'$. For if $\gamma < k$, then, we must have $\hat{x} < \beta < \gamma < k$. But, then, $\gamma \in (\hat{x}, k)$, and so $\beta = h(\gamma)$ implies that $\beta < \hat{x}$, a contradiction. Thus, the claim that $\gamma > k$ is established. But, then, we can infer that $\beta = (1-d)\gamma$.

We claim, next, that $\beta \in B$. Since $\beta \in B \cup C'$, we must have $\beta > k$ if the claim is false. But, then, we can infer that $\alpha = h(\beta) = (1-d)\beta > \hat{x}$, a contradiction, since $\alpha \in A'$. Thus, our claim that $\beta \in B$ is established. Since $\alpha = h(\beta)$, we can also infer that $\alpha \geq (1-d)$.

To summarize our findings so far, we have:

(*i*) $\gamma > k > \beta > \hat{x} > \alpha \geq (1-d)$ and

(*ii*) $\gamma = (1/a) - \xi\alpha;\ \beta = (1-d)\gamma$

Moreover, since, $\beta = (1-d)\gamma$, $\gamma = (1/a) - \xi\alpha$, and $\alpha = h(\beta)$, we have

$$\beta = (1-d)[1/a - \xi h(\beta)] \tag{4}$$

We consider two cases: (i) $\beta \in (\hat{x}, 1)$; (ii) $\beta \in [1, k)$.

We first consider $\beta \in (\hat{x}, 1)$. Then by Proposition 2, $h(\beta) \geq (1/a) - \xi\beta$. By using (4), we have $\beta \leq (1-d)[1/a - \xi(1/a - \xi\beta)]$ which leads to $\beta \geq \frac{(\xi-1)(1-d)}{a[(1-d)\xi^2-1]}$. Since $\beta < 1$, using the equation above we obtain $1 \geq \frac{(\xi-1)(1-d)}{a[(1-d)\xi^2-1]}$ which can be simplified to be $(\xi-1)(1-d) \geq 1$.

We then consider $\beta \in [1, k)$. By Proposition 2, $h(\beta) \geq (1-d)\beta$. By using (4), we have $\beta \leq (1-d)[1/a - \xi(1-d)\beta]$ which leads to $\beta \leq \frac{1-d}{a(1+\xi(1-d)^2)}$. Since $\beta \geq 1$, using the equation above we obtain $1 \leq \frac{1-d}{a(1+\xi(1-d)^2)}$ which again can be simplified to be $(\xi-1)(1-d) \geq 1$. Thus, we have obtained the desired conclusion. ■

With Lemma 5 in hand, we can turn to the proof of Part (ii) of Theorem 1.

**Proof of Theorem 1: Part (ii).** We break up the proof into five separate steps.

**Step 1.** Denote the optimal policy function by $h$, and the 3-period cycle stocks by $\alpha, \beta, \gamma$. Without loss of generality we may suppose that $\alpha < \beta < \gamma$. Following the argument before Eq. (4) in the proof of Lemma 5, we have:

(*i*) $\gamma > k > \beta > \hat{x} > \alpha \geq (1-d)$ and

(*ii*) $\gamma = (1/a) - \xi\alpha;\ \beta = (1-d)\gamma$

**Step 2.** We claim now that $V'_-(\hat{x}) < (a/\rho)$. For, if $V'_-(\hat{x}) \geq (a/\rho)$, then by Lemma 4 there is some $x \in B$, such that $\hat{x} \in h(x)$. And, by (1) of Lemma 1, we then have $\hat{x} \in h(x)$ for all $x \in B$. Since $h$ is a policy function, we must have $h(x) = \hat{x}$ for all $x \in B$. But, we have already noted above that $\beta \in B$, $h(\beta) = \alpha$, and $\alpha < \hat{x}$, so that $\alpha \notin B$. This establishes $V'_-(\hat{x}) < (a/\rho)$.

Our next claim is that $V'_+(\alpha) \leq (a/\rho)$. For suppose $V'_+(\alpha) > (a/\rho)$. Then, we have $V'_+(1-d) > (a/\rho)$, and by (6) of Lemma 1, there exist $m, n$ satisfying $\hat{x} < m < 1 < n < k$, such that $H(x) \in h(x)$ for all $x \in (\hat{x}, m] \cup [n, k)$ and $H(m) = H(n) \in h(x)$ for all $x \in [m, n]$, and since $h$ is a policy function, we have $h(x) \geq H(n)$ for all $x \in B$ using the definition of $H$. Since $\beta \in B$, and $\alpha = h(\beta)$, we must therefore have $\alpha \geq H(n)$. By the concavity of $V$, it follows that $V'_+(H(n)) \geq V'_+(\alpha) > (a/\rho)$. Again, by (6) of Lemma 1, we know $V'_+(H(x)) \leq (a/\rho)$ for all $x \in (\hat{x}, m] \cup [n, k)$, which leads to a contradiction and establishes $V'_+(\alpha) \leq (a/\rho)$.

**Step 3.** Using the fact that $\alpha < \hat{x}$, and $h(\alpha) = \gamma = H(\alpha) > k$, we have $V(\alpha) = \alpha + \rho + \rho^2 V(\beta)$. For $\varepsilon > 0$ and small enough to ensure that $\alpha + \varepsilon < \hat{x}$ and $H(\alpha - \varepsilon) > k$, we also have $V(\alpha + \varepsilon) = (\alpha + \varepsilon) + \rho + \rho^2 V(H^2(\alpha + \varepsilon))$. Thus, we get:

$$\begin{aligned} V(\alpha+\varepsilon) - V(\alpha) &= \varepsilon + \rho^2[V(H^2(\alpha+\varepsilon)) - V(\beta)] \\ &= \varepsilon - \rho^2[V(\beta) - V(\beta - (1-d)\xi\varepsilon)] \end{aligned}$$

Dividing through in the equation above by $\varepsilon$, and then letting $\varepsilon \to 0$, we obtain:

$$V'_+(\alpha) = 1 - \rho^2(1-d)\xi V'_-(\beta) \tag{5}$$

This is the key relationship that will be used in the next Step.

**Step 4.** We now consider two cases separately: (i) $\beta \in (1, k)$; (ii) $\beta \in (\hat{x}, 1]$.

In case (i), define $y = 1 - a[\alpha - (1-d)\beta]$. Then, since $\alpha \geq (1-d)\beta$, we have $y \leq 1 < \beta$. Also, $y > 1 - a\alpha > 1 - a\hat{x} > 0$. Thus, $u(\beta, \alpha) = y$, and since $\alpha = h(\beta)$, we can write $V(\beta) = y + \rho V(\alpha)$. Pick $\varepsilon > 0$ and small enough to ensure that $(\beta - \varepsilon) > 1$, and $y - \varepsilon > 0$. Now, define $y' = y - a(1-d)\varepsilon$. Then, $y' > 0$, and $y' < 1 < (\beta - \varepsilon)$. Also, $y' + a[\alpha - (1-d)(\beta - \varepsilon)] = y' + a[\alpha - (1-d)\beta] + a(1-d)\varepsilon = y + a[\alpha - (1-d)\beta] = 1$. Finally, we have $[\alpha - (1-d)(\beta - \varepsilon)] = [\alpha - (1-d)\beta] + (1-d)\varepsilon > 0$. Thus, $(\beta - \varepsilon, \alpha) \in \Omega$, and $u(\beta - \varepsilon, \alpha) \geq y'$. We can then write $V(\beta - \varepsilon) \geq y' + \rho V(\alpha)$. Thus, we get:

$$\frac{V(\beta) - V(\beta - \varepsilon)}{\varepsilon} \leq a(1-d)$$

Now, letting $\varepsilon \to 0$, we obtain $V'_-(\beta) \leq a(1-d)$. Since we have shown $V'_+(\alpha) \leq (a/\rho)$, using (5), we further obtain:

$$(a/\rho) \geq V'_+(\alpha) = 1 - \rho^2(1-d)\xi V'_-(\beta) \geq 1 - \rho^2 a\xi(1-d)^2$$

which can be simplified to $\rho^3 a\xi(1-d)^2 - \rho + a \geq 0$. Following the proof of Lemma 3, we obtain $\xi(1-d)\rho^2 + \xi\rho - 1 \leq 0$. That is, using the notation introduced in (3), $f(\rho) \leq 0$. This implies:

$$\rho \leq \hat{r} = \frac{-\xi + \sqrt{\xi^2 + 4\xi(1-d)}}{2\xi(1-d)} \tag{6}$$

We will now show that condition (6) also holds in case (ii), that is when $\hat{x} < \beta \leq 1$. It is convenient to subdivide this case into two subcases: (a) $h(\beta) > H(\beta)$, (b) $h(\beta) = H(\beta)$. In case (a), we have $\hat{x} > \alpha = h(\beta) > H(\beta)$. Consequently, if $\varepsilon > 0$ is chosen sufficiently small, it is possible to obtain a terminal stock of $\alpha$, starting from $(\beta - \varepsilon)$. To verify this formally, define $\delta = h(\beta) - H(\beta)$. Note that $0 < \delta < h(\beta) = \alpha < \hat{x} < \beta$. Then, we can write:

$$\begin{aligned} a[\alpha - (1-d)\beta] &= a[H(\beta) + \delta - (1-d)\beta] \\ &= a[(1/a) - \xi\beta + \delta - (1-d)\beta] \\ &= a\delta + a[(1/a) - \xi\beta - (1-d)\beta] \\ &= a\delta + 1 - a\xi\beta - a(1-d)\beta \end{aligned}$$

and $y \equiv 1 - a[\alpha - (1-d)\beta] = a\xi\beta + a(1-d)\beta - a\delta = \beta - a\delta > \beta(1-a) > 0$.

Pick $\varepsilon > 0$ with $\varepsilon$ sufficiently small so that $a(1-d)\varepsilon < y$ and $[1 - a(1-d)]\varepsilon < a\delta$. Defining $y' = y - a(1-d)\varepsilon$, we have $y' > 0$ and $y' = \beta - a\delta - a(1-d)\varepsilon < \beta - \varepsilon$. Further,

$$\begin{aligned} & y' + a[\alpha - (1-d)(\beta - \varepsilon)] \\ &= y - a(1-d)\varepsilon + a[\alpha - (1-d)(\beta - \varepsilon)] \\ &= y + a[\alpha - (1-d)\beta] = 1 \end{aligned}$$

and $[\alpha - (1-d)(\beta-\varepsilon)] = [\alpha - (1-d)\beta + (1-d)\varepsilon] > [\alpha - (1-d)\beta] \geq 0$. Thus, $(\beta - \varepsilon, \alpha) \in \Omega$ and $u(\beta - \varepsilon, \alpha) \geq y' = y - a(1-d)\varepsilon$. We can now write:

$$\left.\begin{aligned} & V(\beta) = y + \rho V(\alpha) \\ & V(\beta - \varepsilon) \geq y' + \rho V(\alpha) = y - a(1-d)\varepsilon + \rho V(\alpha) \end{aligned}\right\}$$

which yields $V'_{-}(\beta) \leq a(1-d)$. Now one can follow exactly the steps in case (i) to obtain condition (6).

We now consider subcase (b) of (ii), where $\hat{x} < \beta \leq 1$ and $h(\beta) = H(\beta)$. In this case, we can write:

$$\left.\begin{aligned} & V(\beta) = \beta + \rho V(\alpha) \\ & V(\beta - \varepsilon) = (\beta - \varepsilon) + \rho V((1/a) - \xi(\beta - \varepsilon)) \end{aligned}\right\}$$

where $\varepsilon > 0$ is small enough to ensure that $\hat{x} < \beta - \varepsilon$. Note that the second line of the equation above follows from (7) of Lemma 1.[16] Then the equation above yields $V'_{-}(\beta) = 1 - \rho\xi V'_{+}(\alpha)$, and using (5), we obtain $V'_{-}(\beta) = (1 - \rho\xi)/[1 - \rho^3\xi^2(1-d)]$. Again, since we have shown $V'_{+}(\alpha) \leq (a/\rho)$, using (5), we obtain:

$$(a/\rho) \geq V'_{+}(\alpha) = 1 - \left[\frac{\rho^2(1-d)\xi(1-\rho\xi)}{1 - \rho^3\xi^2(1-d)}\right]$$

which can be simplified to $\rho^3 a\xi(1-d)^2 - \rho + a \geq 0$. Now one can follow the steps in case (i) to obtain condition (6).

**Step 5.** Consider the following constrained maximization problem, subsequently to be referred to as **CMP**:

$$\left.\begin{array}{ll} \textit{Maximize} & r \\ \textit{subject to} & 1 - sr - s(1-q)r^2 \geq 0 \\ & (1-q) \geq 0 \\ & (s-1)(1-q) - 1 \geq 0 \\ & (q, r, s) \geq 0 \end{array}\right\} \quad (\textbf{CMP})$$

which has a unique solution $(\bar{q}, \bar{r}, \bar{s}) = (0, \hat{\rho}, 2)$.[17] Since the RSS model $(a, d, \rho)$ has an optimal policy function $h$ which generates a period three cycle, we know from Lemma 5 that $(\xi - 1)(1-d) \geq 1$ Thus, using the definition of $\hat{r}$ in condition (6), $(q, r, s) = (d, \hat{r}, \xi)$ satisfies the constraints of the optimization problem above. Since $d > 0$, $(q, r, s) \neq (\bar{q}, \bar{r}, \bar{s})$. Thus, we must have $\hat{r} < \hat{\rho}$, and using Condition (6), we have $\rho < \hat{\rho}$, proving the part (ii) of the Theorem. ■

**Remark 2.** The proof of this theorem does not rely on $h$ being a function. In particular, we can still show that $V'_{+}(\alpha) \leq (a/\rho)$ in Step 2 of the proof of part (ii) if $h$ is not an OPF. To see this, we first claim $V'_{-}(\hat{x}) < (a/\rho)$. Suppose on the contrary $V'_{-}(\hat{x}) \geq (a/\rho)$. Consider $h(\beta)$ is not a singleton: $\alpha, \hat{x} \in h(\beta)$ and $\alpha \neq \hat{x}$. According to (3) and (4) of Lemma 1, $\rho = 1/\xi$. Since we know $\hat{x} \in h(x)$ for all $x \in B$, $V(x) = u(x, \hat{x}) + \rho V(\hat{x})$ for $x \in B$. For $x \in (\frac{k}{(1-d)^n}, \frac{k}{(1-d)^{n+1}}]$ $(n = 0, 1, 2, \ldots)$, $V(x) = \frac{1-\rho^{n+1}}{1-\rho} + \rho^{n+1}V((1-d)^{n+1}x)$. For $x < \hat{x}$, $V(x) = x + \rho V(\frac{1}{a} - \xi x)$. Based on the value function and given $\rho = 1/\xi$, we can show $h(\beta) = [x_0, \hat{x}]$ with $(1/a) - \xi x_0 \leq k$. Since $\gamma > k$, $\alpha < x_0$ and thus $\alpha$ is not optimal, leading to a contradiction. We then claim $V'_{+}(\alpha) \leq (a/\rho)$. For suppose $V'_{+}(\alpha) > (a/\rho)$. We follow the same argument as in the proof and consider instead $\alpha \in h(\beta)$ and $\alpha < H(n)$. Given concavity of $V$, we must have $[\alpha, H(n)] \subset h(\beta)$. Thus, for any $x \in [\alpha, H(n)]$, $V(\beta) = 1 - a(x - (1-d)\beta) + \rho V(x)$, which implies $V'_{+}(\alpha) = (a/\rho)$. This contradicts to the supposition $V'_{+}(\alpha) > (a/\rho)$.

**Proof of Theorem 2.** The straightforward proof of Part (i) is already furnished after the statement of the theorem; and it is also indicated there that the proof of Part (ii) is a direct consequence of Lemma 5. ■

We conclude this subsection with a Proof of Corollary 1.

**Proof of Corollary 1.** The proof follows closely from the Proofs of Propositions 6 and 7 in Khan and Mitra (2020).

Proof of part (i): Let $0 < a < 1/3$. Define

$$d(a) = 1 - \frac{2}{1/a - 1 + \sqrt{(1/a - 1)^2 - 4}}.$$

Since $0 < a < 1/3$, $(1/a - 1)^2 - 4 > 0$. Then, $d(a)$ is a real number. Since $1/a > 1$, $d(a) < 1$. Also, $d(a) > 0$ because $1/a - 1 > 2$. Define $\xi(a) = 1/a - (1 - d(a))$. We have

$$(1 - d(a))(\xi(a) - 1) = 1.$$

Pick $d = d(a)$. Then we have $(1-d)(\xi - 1) = 1$. It then follows from Theorem 2 that there exists $\rho$ such that $(a, d, \rho)$ is an RSS model, which has an optimal policy function that generate a 3-period cycle.

Proof of part (ii): According to Theorem 2, if the optimal policy function generates a 3-period cycle, then $(1-d)(\xi - 1) \geq 1$, which implies

$$\frac{1}{a} \geq \frac{1}{1-d} + (1-d) + 1 > 3 \Longrightarrow a < 1/3.$$

This completes the proof. ■

## Appendix. An optimization problem

This Appendix concerns the optimization problem specified as **CMP** in Step 5 of the proof of Part (ii) of Theorem 1. Define the constraint set of **CMP** as

$$\begin{aligned} C = \{&(q, r, s) \geq 0 : 1 - sr - sr^2(1-q) \geq 0, \\ &(1-q) \geq 0, (s-1)(1-q) - 1 \geq 0\} \end{aligned}$$

Note that $C$ is not bounded above in $s$, and so Weierstrass' theorem cannot be applied to ensure the existence of a solution to **CMP**. However, the following variation of Weierstrass theorem can be used. Define:

$$C' = \{(q, r, s) \in C : s \leq 3\}$$

Note that $(\bar{q}, \bar{r}, \bar{s}) = (0, \hat{\rho}, 2)$ belongs to $C'$, where $\hat{\rho} = [(\sqrt{3} - 1)/2]$. Further, $C'$ is a closed and bounded subset of $\mathbb{R}^3$. Thus,

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[16] For some monotone properties of the optimal policy function see Lemma 4 – 6 in Khan and Mitra (2020).

[17] For details of the solution, see Appendix.

by Weierstrass theorem, there is a solution $(q'', r'', s'')$ to the problem:

$$\left.\begin{array}{ll} \textit{Maximize} & r \\ \textit{subject to} & (q,r,s) \in C' \end{array}\right\}$$

Thus, for all $(q, r, s) \in C'$, we have $r \leq r''$. And, for $(q, r, s) \in C \backslash C'$, we have $s > 3$, and so by the first constraint in $C$, we must have $r < (1/3)$. Thus, $r < \hat{\rho} \leq r''$, since $\hat{\rho} > 0.36$. To summarize, for all $(q, r, s) \in C$, we must have $r \leq r''$, and so $(q'', r'', s'')$ solves problem **CMP**.

Now consider an arbitrary solution to problem **CMP**, and call this $(q', r', s')$. We will show that $(q', r', s')$ must be equal to $(\bar{q}, \bar{r}, \bar{s}) = (0, \hat{\rho}, 2)$. This will establish that $(0, \hat{\rho}, 2)$ is the *unique* solution to problem **CMP**. Note that since $(q', r', s') \in C$, we must have $q' < 1, s' > 1, r' < 1$, and since $(q', r', s')$ solves **CMP**, while $(\bar{q}, \bar{r}, \bar{s}) = (0, \hat{\rho}, 2) \in C$, we must also have $r' \geq \hat{\rho} > 0$.

(a) We claim first that the third constraint must be binding:

$$(s' - 1)(1 - q') = 1 \tag{7}$$

For if $(s'-1)(1-q') > 1$, then we can define $(q, r, s) = (q', r', s' - \varepsilon)$, where $\varepsilon > 0$ is sufficiently small so that $((s'-\varepsilon)-1)(1-q') > 1$. Then the second and third constraints are satisfied at $(q, r, s)$. Further,

$$\begin{aligned} sr + sr^2(1-q) &= s'r' + s'(r')^2(1-q') - \varepsilon r' - \varepsilon(r')^2(1-q') \\ &< s'r' + s'(r')^2(1-q') \leq 1 \end{aligned}$$

So, it is now possible to define $(\tilde{q}, \tilde{r}, \tilde{s})$ with $\tilde{q} = q, \tilde{s} = s$, and $\tilde{r} > r = r'$ with $\tilde{r}$ sufficiently close to $r$ so that $(\tilde{q}, \tilde{r}, \tilde{s})$ satisfies all three constraints of **CMP**. But this contradicts the optimality of $(q', r', s')$.

(b) We claim next that the first constraint must be binding:

$$s'r' + s'(r')^2(1 - q') = 1 \tag{8}$$

For if $s'r' + s'(r')^2(1 - q') < 1$, then one can define $(q, r, s) = (q', r' + \varepsilon, s')$ with $\varepsilon > 0$ sufficiently small so that the first constraint is satisfied. Further, the second and third constraints are clearly satisfied. But this contradicts the optimality of $(q', r', s')$.

(c) We claim now that:

$$s' = 2 \tag{9}$$

Note that by the third constraint, we must have $s' \geq 2$. So, if (9) is not satisfied, we must have $s' > 2$.

For any $s \in (2, s')$ we can define $1 - q = 1/(s - 1)$. Then $0 < q < 1$, and the second and the third constraints are satisfied. Also, using (7), and the convexity of the function $[s/(s - 1)]$ for $s > 1$, we get:

$$\begin{aligned} &[sr' + s(r')^2(1-q)] - [s'r' + s'(r')^2(1-q')] \\ &= [sr' + \frac{s(r')^2}{(s-1)}] - [s'r' + \frac{s'(r')^2}{(s'-1)}] \\ &\leq (s-s')r' + (s-s')(r')^2\frac{(-1)}{(s-1)^2} \\ &= (s-s')r'[1 - \frac{r'}{(s-1)^2}] < 0 \end{aligned} \tag{10}$$

the last line of (10) following from the facts that $s > 2$, while $r' < 1$, and $(s-s') < 0$. Thus, $(q, r', s)$ satisfies the first constraint with strict inequality

$$sr' + s(r')^2(1 - q) < [s'r' + s'(r')^2(1 - q')] = 1.$$

But then, as in part (b) above, one can define $(\tilde{q}, \tilde{r}, \tilde{s}) = (q, r' + \varepsilon, s)$ with $\varepsilon > 0$ sufficiently small so that the first constraint is satisfied. Further, the second and third constraints are clearly satisfied. But this contradicts the optimality of $(q', r', s')$. This establishes claim (9).

(d) Using (7) and (9), we get $q' = 0$. Using this, along with (8) and (9), we get $2r' + 2(r')^2 = 1$. Then, using the fact that $r' > 0$, we get $r' = \hat{\rho}$, completing the proof. ■

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