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# Continuous unimodal maps in economic dynamics: On easily verifiable conditions for topological chaos ☆

Liuchun Deng [a,*], M. Ali Khan [b], Tapan Mitra [c]

[a] Social Sciences Division, Yale-NUS College, Singapore
[b] Department of Economics, The Johns Hopkins University, Baltimore, MD 21218, United States of America
[c] Department of Economics, Cornell University, Ithaca, NY 14853, United States of America

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**Abstract**

In this paper, we offer necessary and sufficient conditions for the presence of *odd-period cycles* and *turbulence* in a continuous unimodal interval map. The characterizations we present are original both to the economic and the mathematical literature, and go beyond existential assertions to easy verifiability. We apply our two theorems to six different canonical models in the literature on economic dynamics, all being grounded in the fact that their policy functions are given by continuous unimodal maps. An unintended outcome of the work presented here is to alert the economics profession to a richer conception of erratic and chaotic dynamics, one that goes considerably beyond the 1964-1975 Sharkovsky-Li-Yorke emphasis on three-period cycles and on uncountable scrambled sets.


☆ This project reports results obtained with Tapan Mitra during the last five months of his life: he played an indispensable role in bringing them to fruition, and as such it is joint work in the full and total sense of the term. In polishing up the work for publication, both of the surviving authors have been in silent but constant conversation with him: checking the moves that he would have approved and others he would not have. The responsibility for any blemishes in this final version is however solely theirs. The authors thank Fabio Privileggi and Gerhard Sorger for advice and instruction on what was the driving thrust of Tapan Mitra's work in capital theory and economic dynamics as well as in the subject of growth and development, and Marco Battaglini, Jess Benhabib, Minako Fujio, Oded Galor, Andreas Irmen, Ka-Kit Iong, Debraj Ray and Santanu Roy for stimulating correspondence. The introduction of this work has also been inspired by Kazuo Nishimura's brilliant *Lionel McKenzie Lecture* delivered to the *Society for the Advancement of Economic Theory* at its $21^{st}$ virtual meeting on June 14th, 2021. Liuchun Deng acknowledges the support of the Start-up Grant from Yale-NUS College.

* Corresponding author.
*E-mail address:* liuchun.deng@yale-nus.edu.sg (L. Deng).

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**Contents**



> The notion of chaos does not come from mathematics. It was developed by physicists. Afterwards, it gradually spread to computer science and other fields, where it took on slightly different meanings. When writing about chaos, a mixture of mathematics and empirical ideas cannot be avoided. Many properties are called chaotic, but none can be *proved* to be chaotic because there is no definition of chaos and there will not be any.[1] Blanchard (2009)

## 1. Introduction

Since the introduction of "erratic growth" by Benhabib and Day (1980), and later followed up by Day and Shafer (1985) and Nishimura and Yano (1996), there is by now a good professional understanding of chaotic dynamics, and of the distinction between the subject's topological and ergodic (statistical) aspects.[2] However, as far as the topological aspect is concerned, this understanding is largely grounded in the sufficient condition on the existence of *three-period*

[1] The italics and sentences are the author's but we have re-arranged them to emphasize that in this work we elaborate on, and investigate, only one possible definition of chaos. Blanchard lists five types of chaos and labels them as (i) sensitivity, (ii) positive entropy, (iii) Li-Yorke, (iv) Auslander-Yorke, and (v) Devaney. In this paper, we rely on (ii) as opposed to the reliance on (iii) in economic dynamics. The interested reader is also referred to Kolyada and Snoha (1997), Brown and Chua (1996, 1998) and above all to Ruette (2017) for additional definitions of chaos.

[2] For surveys of this literature, see Benhabib's (2008) Palgrave entry on "chaotic dynamics"; also see Benhabib (1992), Majumdar et al. (2000), Dana et al. (2006), and Bhattacharya and Majumdar (2007). In Deng and Khan (2017), the authors present a portmanteau theorem that shows the equivalence of several notions of chaos culled from Ruette's authoritative survey.

*cycles* furnished by Li and Yorke (1975). To be sure, even though Mitra (2001) went beyond the Sharkovsky order and the Li-Yorke condition that it implies to introduce notions of *turbulence* and *topological entropy* in the context of Matsuyama's (1999) model, these ideas have remained somewhat alien than they ought to be in terms of application and use.[3] This paper is addressed to this weakness in the literature of economic dynamics.

The problematic addressed in this paper is best begun by a discussion of *topological entropy* in the language of the everyday. Already in his 1995 text, Clark Robinson noted that topological entropy $h(f)$ of a map $f$ is a non-negative real number, and while "it has a complicated definition, it can be thought of as a quantitative measurement of the amount of sensitive dependence on initial conditions of the map. In fact, it is determined by how many 'different orbits' there are for a given map (or flow)."[4] In yet another text, Brucks and Bruin (2004) elaborate on the idea of two orbits being "essentially different," and they focus on the "resolution" of this difference as depending on an arbitrary positive small number $\epsilon$. In sum, topological entropy involves the number of $\epsilon$-essentially different orbits, and this number grows exponentially fast for a map with positive topological entropy.[5] It attests to the instability and unpredictability (its "chaoticity" so to speak) that is inherent in the system. A map is said to exhibit topological chaos if it has positive topological entropy. With this intuitive and informal background of topological entropy at hand, we turn to the introduction of our results.

In this paper, we report two theorems on topological chaos when a map may not admit a three-period cycle and thus the well-known Li-Yorke criterion does not apply. For a class of unimodal interval maps, we provide necessary and sufficient conditions for two particular forms of topological chaos: the existence of odd-period cycles and turbulence of the second iterate of a map. The conditions involve the third iterate of the critical point, the interior fixed point, and benchmarks related to two-period cycles, if any, all of which can be easily computed and verified. Specifically, the conditions do not involve any global properties of the map, and thus go substantially beyond existential assertions to easy verifiability.

Our results build upon the seminal work of Mitra (2001) that identifies for the first time an easily verifiable condition for topological chaos.[6] His condition is a sufficient condition for turbulence of the second iterate of a map. In a recent paper, Deng and Khan (2018) extend this sufficient condition to accommodate a knife-edge case of the third iterate of the critical point coinciding with the interior fixed point. The conditions we identify in this paper contribute to the existing literature in two ways. First, our conditions are both necessary and sufficient, which yield a satisfactory characterization of two particular forms of topological chaos. Second, our conditions cover not only turbulence of the second iterate but also odd-period cycles. Since odd-period cycles and turbulence of the second iterate occupy two consecutive positions in the refined

3 We note for the reader that the notion of *scrambled sets* was original to Li and Yorke (1975) and was not available in Sharkovsky (1964).

4 See page 84 for the first sentence and page 333 for the second in Robinson (1995).

5 In a slightly more formal vernacular, one that emphasizes the adjective *topological* instead of the noun *entropy*, one can develop the notion of topological entropy through a tripartite procedure: to define topological entropy of a *cover* of the domain of a continuous map, then that of a map relative to a *cover*, and finally, the topological entropy of the map as the supremum over all *covers*; see Chapter 3 in the 1998 student text of Pollicott and Yuri. The reader can also look forward to Footnote 10.

6 Easy verifiability is one of the signatures of the later work by Tapan Mitra; also see Mitra and Roy (2017, 2022).

Sharkovsky order, the necessary and sufficient conditions and their evident resemblance are a testimony to the importance of the work of Block and Coppel.[7]

And now we turn from the theorems themselves to the use of these theorems in economic dynamics. Towards this end, we turn to what we see as six paradigmatic models devoted to six different considerations arising in economic theory, and all leading to continuous unimodal maps from an interval to itself. It is remarkable how those maps arise as an instrument of synthesis from considerations of political economy, labor supply, innovation and industrial organization, and development policy.[8] We obtain neat parametric conditions for complicated dynamics for each of the six models, underscoring the easy verifiability of our characterization results. This is done in Section 3. In Section 2, we present the main theoretical results of the paper, the easily verifiable necessary and sufficient conditions for odd-period cycles and turbulence of the second iterate. We conclude the paper with some open questions in Section 4 and relegate all the proofs to Section 5. Any reader interested in working through them will note that they require sustained argumentation.

## 2. The results: necessary and sufficient conditions

### 2.1. Preliminaries

We say that $f$ is an interval map if it is a continuous map from a non-degenerate compact interval $X$ to itself. A point $x \in X$ is called a *periodic point* of $f$ if there exists a positive integer $n$ such that $f^n(x) = x$. An interval map $f$ is said to admit an *n-period cycle* if there exists $x \in X$ such that $n$ is the smallest positive integer for which $f^n(x) = x$ holds. An interval map $f$ is said to admit an *odd-period cycle* if the map admits an $n$-period cycle for some odd number $n > 1$. Denote by $\mathcal{P}_n$ the set of all the interval maps that admit an $n$-period cycle. A map $f$ is said to be *turbulent* if there exist three points, $x_1$, $x_2$, and $x_3$, in $X$ such that $f(x_2) = f(x_1) = x_1$ and $f(x_3) = x_2$ with either $x_1 < x_3 < x_2$ or $x_2 < x_3 < x_1$.[9] Denote by $\mathcal{T}_n$ the set of all the interval maps whose $n$-th iterate is turbulent. We have the following refined Sharkovsky order due to Block and Coppel (1986).

**Theorem BC.** *For a non-degenerate compact interval, the following stratification holds with the inclusions being strict:*

$$\mathcal{T}_1 \subset \mathcal{P}_3 \subset \mathcal{P}_5 \subset \mathcal{P}_7 \subset \cdots \subset \mathcal{T}_2 \subset \mathcal{P}_{3\cdot 2} \subset \mathcal{P}_{5\cdot 2} \subset \mathcal{P}_{7\cdot 2} \subset \cdots \subset \mathcal{T}_4 \subset \mathcal{P}_{3\cdot 4} \subset \mathcal{P}_{5\cdot 4} \subset \mathcal{P}_{7\cdot 4} \\ \subset \cdots \subset \mathcal{T}_{2^n} \subset \mathcal{P}_{3\cdot 2^n} \subset \mathcal{P}_{5\cdot 2^n} \subset \mathcal{P}_{7\cdot 2^n} \subset \cdots \subset \mathcal{P}_4 \subset \mathcal{P}_2 \subset \mathcal{P}_1.$$

[7] See Block and Coppel (1986). For the emphasis on two-period cycles, see Coppel (1955) and its application in Benhabib and Nishimura (1979a,b) and Mitra and Nishimura (2005).

[8] See in sequence Battaglini (2021), Iong and Irmen (2021), Matsuyama (1999), Baumol and Wolff (1992), Deneckere and Judd (1992), and Khan and Mitra (2005b).

[9] This definition of turbulence is originally due to Block and Coppel (1986) and is explored and exposited in greater detail in their book (Block and Coppel, 1992); also see Du (2013) for recent work on the existence of turbulence in the sense of Block-Coppel for a class of interval maps. There is a related but different definition of turbulence under which a map $f$ is said to be turbulent if $f^n$ is turbulent in the sense of Block-Coppel for some positive integer $n$; see Block (1978) and Lasota (1979). For further discussions on the terminology of turbulence, also see Remark 3.28 in Ruette (2017). In this paper, we stick to the definition of turbulence as in Block and Coppel (1986) because this definition of turbulence facilitates the presentation of a delicate stratification result, from which our main characterization results stem; see Theorem BC. We are grateful to an anonymous referee for insisting on this point.

A set $S \subset X$ is said to be a *scrambled set* if for any distinct points $x, y \in S$, we have

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0 \text{ and } \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0,$$

and for $x \in S$ and $y$ being a periodic point of $f$,

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0.$$

With these formalities laid on the table, we turn to the operational definition of topological chaos adopted in Mitra (2001). For a natural number $n$ and a positive real number $\varepsilon$, a finite set $E \subset X$ is called $(n, \varepsilon)$-*separated*, if for every $x, y \in E$, $x \neq y$, there is an integer $k : 0 \leq k < n$ such that $|f^k(x) - f^k(y)| \geq \varepsilon$. Let $s(n, \varepsilon)$ denote the maximal cardinality of an $(n, \varepsilon)$-separated set. We define the topological entropy of an interval map $f$ as

$$\psi(f, I) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} (1/n) \log s(n, \varepsilon).$$

An interval map $f$ is said to exhibit *topological chaos* if it has positive topological entropy.[10] We now turn to the following result.

**Theorem M.** *The topological entropy of an interval map $f$ is positive if and only if $f$ admits a periodic cycle whose period is not a power of* 2.

Theorem M, which is originally due to Misiurewicz (1979, 1980), provides a tight connection between topological entropy and periodic cycles. With this theorem, one can go considerably beyond the three-period cycles to establish the existence of topological chaos. In particular, from Theorem BC, if the second iterate of a map is turbulent (an element of $\mathcal{T}_2$), then this map admits a six-period cycle, and thus, according to Theorem M, exhibits topological chaos. Indeed, we will explore two alternative criteria for topological chaos: odd-period cycles and turbulence of the second iterate, the two "adjacent" classes of interval maps in Block and Coppel's refined Sharkovsky order. It is known that if a map admits a cycle whose period is not a power of 2, then this map has an uncountable scrambled set.[11] From Theorem M, this further implies that if a map exhibits topological chaos, it has an uncountable scrambled set.[12]

From now on, we restrict our attention to a family of unimodal interval maps. Let $\mathcal{F}$ be the set of continuous maps from an interval, $[a, b]$, to itself, with a generic element $f$, satisfying the following two conditions:

(i) There is $m$ in (a,b), with the map $f$ strictly increasing on $[a, m]$ and strictly decreasing on $[m, b]$.

[10] With this definition at hand, we can turn to what we earlier referred to in Footnote 5 as the topologist's language of the everyday that concerns the tripartite procedure pertaining to the covers of the domain of a continuous map. What now needs to be pinned down is that these two definitions of topological entropy unambiguously lead to the same concept. This is Proposition 3.8 in Pollicott and Yuri (1998), and the tripartite procedure is quantitatively executed in the definitions and formulae in their Section 3.1: we refer the interested reader to these pages.

[11] See Proposition 27 in Chapter VI of Block and Coppel (1992).

[12] However, it should be noted that an uncountable scrambled set can still be of measure zero, and our results do not speak about the well-known issue of the (un)observability of chaos. Chaotic maps can give rise to rather predictable dynamics; see Khan and Rajan (2017) as an extension of Nathanson (1976). We thank a referee for his/her emphasis on this.

(ii) $f(a) \geq a$, $f(b) < b$, and $f(x) > x$ for all $x$ in $(a, m]$.

There is a unique interior fixed point of $f$, which is in $(m, b)$, denoted by $z$. We must have $f(x) > x$ for all $x$ in $(a, z)$, and $f(x) < x$ for all $x$ in $(z, b]$. We define the two sets: $I \equiv [a, m]$; $D \equiv [m, b]$. Given unimodality, the $I$, $D$ notation emphasizes that the function is increasing on the "left" interval and decreasing on the "right" one. And to be sure, this is without any loss of generality in that the results presented below transpose easily as a consequence of unimodality to the case where the function is decreasing on the left interval and increasing on the right one, and the map has a sink rather than a peak.

Define $\Pi \equiv \{x \in [m, b] : f(x) \in [m, b] \text{ and } f^2(x) = x\}$. This is the set of the fixed point and period-two points, *entirely* confined to the set $D$. There is no presumption that $\Pi$ is a finite set.[13] The set $\Pi$ is nonempty since there always exists a fixed point in the interior of $D$, and it belongs to $\Pi$. By the continuity of $f$, $\Pi$ is closed. Hence, even if $\Pi$ is infinite, $\bar{\mu} \equiv \max\{x \in \Pi\}$ and $\underline{\mu} \equiv \min\{x \in \Pi\}$ are well-defined and belong to $\Pi$. To ease the presentation and exposition of our results, we define the following conditions:

**Condition OPC**: The function $f$ satisfies $f^2(m) < m$ and $f^3(m) < \underline{\mu}$.

**Condition TUR**: The function $f$ satisfies $f^2(m) < m$ and $f^3(m) \leq \bar{\mu}$.

Naming of the two conditions corresponds to the dynamics that they will imply. We will later establish that Condition OPC is necessary and sufficient for the existence of odd-period cycles and Condition TUR is necessary and sufficient for turbulence of the second iterate. In the special case that there is no 2-period cycle entirely confined to the set $D$, $\Pi$ is a singleton $\{z\}$ with $\underline{\mu} = \bar{\mu} = z$, and so both conditions are stated in the case of the fixed point $z$. We shall then refer to these conditions as OPC$^z$ and TUR$^z$.

### 2.2. *Characterization of odd-period cycles*

Cycles of odd period $n > 1$ play a very important role in the theory of topological chaos. They also constitute a striking part of the Sharkovsky order. In this subsection, we provide a complete characterization of the odd-period cycles for the class of unimodal maps $\mathcal{F}$. The sufficient condition for odd-period cycles can be easily obtained through the LMPY theorem in Li et al. (1982). We focus on the necessary condition for odd-period cycles. It is more convenient to establish the necessary condition first for the case of an $n$-period cycle with $n > 1$ and $n$ odd, where there is no periodic cycle of odd period strictly between 1 and $n$. These are called *Stefan cycles* (Stefan, 1977), and they have a particularly rigid structure that one can exploit.[14] We then extend the result to all odd-period cycles.

**Lemma 1.** *If $f \in \mathcal{F}$ has a periodic cycle of odd period $n \geq 3$, but no periodic cycle of odd period strictly between 1 and $n$, then $f^j(m) \leq m$ for some odd $j$ with $3 \leq j \leq n + 2$.*

**Theorem 1.** *Let $f \in \mathcal{F}$. The map $f$ admits an odd-period cycle if and only if $f^k(m) \leq m$ for some odd $k > 1$.*

[13] Example 1 presents a situation in which there are a continuum of two-period cycles.

[14] Note that three-period cycles are always Stefan cycles, but not all five-period cycles are Stefan cycles; see Robinson (1995).

The elegance of Theorem 1 notwithstanding, it is an existential claim and hence not easily verifiable. The question then remains whether there is an easily verifiable necessary and sufficient condition for the existence of odd-period cycles. Towards that, we offer Theorem 2 which identifies Condition OPC as the condition one is after. Moreover, in the economic applications in what follows, $\Pi$ is a singleton and so the result can be further sharpened to a requirement hinging only on the critical and the fixed points.

**Theorem 2.** *Let $f \in \mathcal{F}$. The map $f$ admits an odd-period cycle if and only if Condition OPC is satisfied.*

**Corollary 1.** *Let $f \in \mathcal{F}$ with $\Pi$ being a singleton. The map $f$ admits an odd-period cycle if and only if Condition OPC$^z$ is satisfied.*

### 2.3. Characterization of turbulence of the second iterate

We now provide the necessary and sufficient condition for the second iterate of a map being turbulent. Again, if $\Pi$ is a singleton, the condition we identify can be further sharpened as in Corollary 2.

**Theorem 3.** *Let $f \in \mathcal{F}$. The second iterate $f^2$ is turbulent if and only if Condition TUR is satisfied.*

**Corollary 2.** *Let $f \in \mathcal{F}$ with $\Pi$ being a singleton. The second iterate $f^2$ is turbulent if and only if Condition TUR$^z$ is satisfied.*

Theorem 3 and its corollary strengthen the main result in Mitra (2001) and its recent generalization. In Mitra (2001), the main characterization result, Proposition 2.3, states that for $f \in \mathcal{F}$, if $f^2(m) < m$ and $f^3(m) < z$, then $f^2$ is turbulent. Deng and Khan (2018) generalize this sufficient condition for $f^2$ being turbulent by replacing $f^3(m) < z$ with $f^3(m) \leq z$. In contrast, this paper provides a necessary *and* sufficient condition for the second iterate being turbulent. For the special case of $\Pi$ being a singleton, Corollary 2 demonstrates the sufficient condition identified in Deng and Khan (2018) to be also necessary.

Moreover, according to Corollaries 1 and 2, if $\Pi$ is a singleton, Condition OPC can be simplified as $f^2(m) < m$ and $f^3(m) < z$ and Condition TUR boils down to $f^2(m) < m$ and $f^3(m) \leq z$. In this case, the difference between the existence of the odd-period cycles and turbulence of the second iterate exactly lies in the borderline case for $f^3(m) = z$, which is the focus of the "easy extension" as in Deng and Khan (2018), thereby highlighting that the borderline case is, albeit not being robust to small parametric perturbation, conceptually important.

However, if $\Pi$ is not a singleton and therefore $\underline{\mu} < \bar{\mu}$, then the distinction between odd-period cycles and the turbulence of the second iterate becomes more "visible". To illustrate our main results for a non-singleton $\Pi$, we consider the following piece-wise linear map $f$ from [0, 1] to itself:

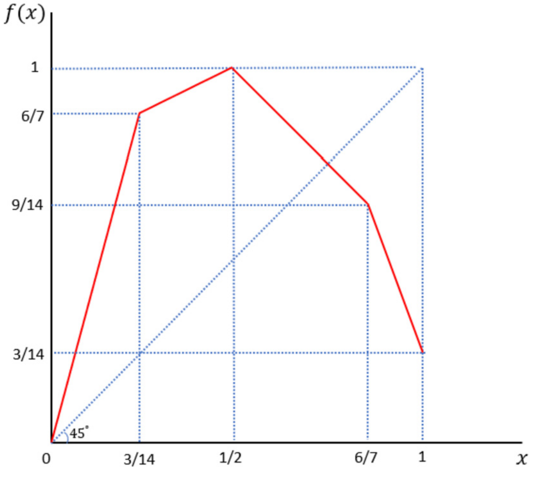


Fig. 1. The Map for Example 1.

**Example 1.**

$$f(x) = \begin{cases} 4x & \textit{for } 0 \le x \le 3/14 \\ 6/7 + (1/2)(x - 3/14) & \textit{for } 3/14 < x \le 1/2 \\ 1 - (x - 1/2) & \textit{for } 1/2 < x \le 6/7 \\ 9/14 - 3(x - 6/7) & \textit{for } 6/7 < x \le 1. \end{cases}$$

The map is plotted in Fig. 1. The modal point $m = 1/2$. The interior fixed point $z = 3/4$. There is a two-period cycle in the downward sloping part of the map, with the periodic points 6/7 and 9/14. In fact, there are a continuum of two-period cycles with $\Pi = [9/14, 6/7]$. The first four iterates of $f$ starting from $m$ can be calculated as follows

$$f(m) = 1;\; f^2(m) = 3/14;\; f^3(m) = 6/7;\; f^4(m) = 9/14.$$

Clearly, $f^2(m) < m$ but $f^3(m) > z$, so the original theorem in Mitra (2001) does not apply. However, since $f^3(m) = 6/7 \in \Pi$, Theorem 3 suggests $f^2$ is turbulent. This is indeed the case, which can be seen from $f^2(3/14) = 9/14 = f^2(9/14)$, $f^2(1/2) = 3/14$, and $3/14 < 1/2 < 9/14$. Further, since $f^3(m) = 6/7 \ge \underline{\mu} = 9/14$, $f$ does not admit an odd-period cycle.

A natural question arises: given the usefulness of turbulence, why do we focus only on the second iterate? According to the refined Sharkovsky order in Block and Coppel (1986), turbulence of the map itself precedes the three-period cycle. In some of the economic applications, we already know that the map does not have a three-period cycle (Matsuyama, 1999; Khan and Mitra, 2005b) and thus it precludes the map itself from being turbulent. Moreover, assuming that we do not have any prior information about the existence of a three-period cycle of the map, the following result suggests that turbulence of the map itself leads to a very restrictive condition which is unlikely to be satisfied in most economic applications.

**Proposition 1.** *If $f \in \mathcal{F}$ is turbulent, then $f(b) = f(a) = a$ and $f(m) = b$.*

In terms of the refined Sharkovsky order, if we want an easily verifiable condition for topological chaos by exploiting the concept of turbulence, we can then check for turbulence of $f^n$ with $n$ being a power of 2. Clearly, the easiest to check would be turbulence of $f^2$.

Before turning to the applications, let us also note that in some economic models, the dynamics are represented by a unimodal map with a local minimum instead of a local maximum. With some adaptation of the conditions, our results hold for a unimodal map with a local maximum or with a local minimum. Let $\mathcal{G}$ be the set of continuous maps from an interval, $[a, b]$, to itself such that its generic element $g$ satisfying the following two conditions:

(i) There is $m$ in (a,b), with the map $g$ strictly decreasing on $[a, m]$ and strictly increasing on $[m, b]$.
(ii) $g(a) > a$, $g(b) \leq b$, and $g(x) < x$ for all $x$ in $[m, b)$.

There is also a unique interior fixed point $z \in (a, m)$. For $g \in \mathcal{G}$, let $\Pi' \equiv \{x \in [a, m] : g(x) \in [a, m] \text{ and } g^2(x) = x\}$. Like $\Pi$, by construction, we have $z \in \Pi'$. Our main theorems, Theorems 2 and 3, can be easily extended to this "overturned" class of unimodal maps $\mathcal{G}$.

**Corollary 3.** *Let $g \in \mathcal{G}$. The map $g$ has an odd-period cycle if and only if $g^2(m) > m$ and $g^3(m) > \max\{x \in \Pi'\}$ and the second iterate $g^2$ is turbulent if and only if $g^2(m) > m$ and $g^3(m) \geq \min\{x \in \Pi'\}$.*

## 3. Models of intertemporal resource allocation

In this section, we consider a wide array of dynamic economic models which all give rise to unimodal maps as the representation of the dynamics. The first four models yield maps in $\mathcal{F}$ with a local maximum in a dynamic environment with strategic considerations (Battaglini, 2021), in an overlapping-generation setting (Iong and Irmen, 2021), or under an equilibrium growth framework (Matsuyama, 1999; Baumol and Wolff, 1992). The last two models yield maps in $\mathcal{G}$ with a local minimum in both equilibrium and optimum growth settings (Deneckere and Judd, 1992; Khan and Mitra, 2005a). However, it is worth underscoring that our perusal of these papers is primarily motivated by the synthetic thrust of this work: to show the relevance of our two theoretical results pertaining to the existence of *odd-period cycles* and to the phenomena of *turbulence*. It is not intended to substitute for the substantive contributions of the other papers themselves, and the reader is strongly encouraged to go directly to them, though perhaps keeping our summary statements in mind.

### *3.1. The Battaglini model: time inconsistency in environmental protection*

To investigate the inherent unpredictability in dynamic social problems, Battaglini (2021) considers a political economy model of environmental protection. Time inconsistency in policy making arises from the fact that two parties stochastically alternate in power, with each incumbent party under the strong temptation to cater to its own political constituency. Due to time inconsistency, policy making in the model hinges on the expectation of the policy chosen by the future incumbent: the incumbent has lower incentives to pollute if the future incumbent is expected to be a heavy polluter while the incentives to pollute are higher if the future incumbent is expected to adopt a more environmentally-friendly policy. Battaglini (2021) demonstrates time inconsistency to be a novel source of complicated dynamics in the equilibrium evolution of the pollution level and this insight can be applied to a much broader range of models featuring time inconsistency in decision making.

According to Proposition 1 in Battaglini (2021), if the temptation of the incumbent to cater to its constituency is sufficiently strong, then the dynamics of the pollution level $x$ in a symmetric Markov perfect equilibrium can be described as

$$f_b(x) = \phi_1 x - \phi_2 x^2 + c, \text{ for } x \in X_b \equiv \left[\frac{\phi_1 - 1 - \sqrt{(\phi_1 - 1)^2 + 4\phi_2 c}}{2\phi_2}, \frac{\phi_1^2}{4\phi_2} + c\right],$$

where $\phi_1$ and $\phi_2 > 0$ are composite parameters of the model and $c$ arises from the expectation of the future behavior of the other party, satisfying $4 \le 4\phi_2 c + (\phi_1 - 1)^2 \le 9$. Moreover, for any $x \in X_b$, $f(x) \in X_b$ and for any $x \notin X_b$, the economy converges in finite periods to $X_b$. Since $f_b$ is a unimodal map satisfying all the conditions for $\mathcal{F}$, we can identify the critical point $m = \phi_1/(2\phi_2)$ and the interior fixed point $z = (\phi_1 - 1 + \sqrt{(\phi_1 - 1)^2 + 4\phi_2 c})/(2\phi_2)$. As shown by Battaglini (2021), $f_b$ is topologically conjugate to the logistic map $f_\eta(x) = \eta x(1 - x)$ for $x \in [0, 1]$ with $\eta = 1 + \sqrt{(\phi_1 - 1)^2 + 4\phi_2 c}$. Appealing to the results on the logistic map, he establishes the existence of cycles of any period and the occurrence of topological chaos. Our theoretical results enable a more direct way to characterize two specific forms of topological chaos: odd-period cycles and turbulence of the second iterate. By showing that $\Pi$ is a singleton when $f_b^2(m) < m$, we can apply Theorems 2 and 3 to obtain the following proposition.

**Proposition 2.** *Let $\hat{t}$ be the unique real root of the equation $t^3 + t^2 - 5t - 13 = 0$ ($t \approx 2.6786$). Then $f_b$ has an odd-period cycle if and only if $\hat{t}^2 < 4\phi_2 c + (\phi_1 - 1)^2 \le 9$ and $f_b^2$ is turbulent if and only if $\hat{t}^2 \le 4\phi_2 c + (\phi_1 - 1)^2 \le 9$.*

Since $\eta = 1 + \sqrt{(\phi_1 - 1)^2 + 4\phi_2 c}$, the two conditions we identify can further be rewritten as $\hat{t} + 1 < \eta \le 4$ and $\hat{t} + 1 \le \eta \le 4$, respectively. Interestingly, the lower bound $(\hat{t} + 1) \approx 3.6786$ is also identified by Ruelle (1977) as the value for $\eta$ in the logistic map under which there exists an invariant measure.[15] Moreover, since both $\phi_1$ and $\phi_2$ increase with the incumbent's temptation to abuse power in the model, the conditions in Proposition 2 suggest that for chaos to arise, the level of temptation can be neither too high nor too low.

### *3.2. The Iong-Irmen model: the role of declining working hours*

In an overlapping generation setting with exogenous population growth, Iong and Irmen (2021) endogenize both individual supply of working hours and technological progress so as to study the decline of working hours and its role in the emergence of economic fluctuations. There are two growth regimes in equilibrium. If productivity, measured by the number of varieties of the consumption goods, is high relative to population size, then the research sector is inactive and there is no productivity growth. If productivity is low relative to population size, then the research sector comes alive and new varieties will be created. The economy may bounce between the two growth regimes with the dynamics of the labor force to productivity ratio $x$ being described by

15 There is no coincidence because $\eta = \hat{t} + 1$ in Ruelle (1977) is chosen such that $f_\eta^3(m) = z$ holds. Also see Brucks and Bruin (2004) or Robinson (1995).

$$f_{ii}(x) \equiv \begin{cases} (1+g)x & 0 \le x \le x_c \\ (1+g)x_c^{n+1}x^{-n} & x_c \le x \le x_c(1+g), \end{cases}$$

where $g > 0$ is the exogenous growth rate of the labor force, $x_c > 0$ is the critical value of $x$, and $n > 1$ is a composite parameter which decreases with intertemporal knowledge spillover and, more importantly, with the equilibrium elasticity of individual supply of working hours to change in productivity. The positive steady state becomes unstable when $n$ is greater than one, and we focus on this case. The map $f_{ii}$ belongs to $\mathcal{F}$ so we can apply Theorems 2 and 3 to obtain the following simple conditions for topological chaos.

**Proposition 3.** *The map $f_{ii}$ has an odd-period cycle if and only if $n > (\sqrt{5}+1)/2$ and $f_{ii}^2$ is turbulent if and only if $n \ge (\sqrt{5}+1)/2$.*

This proposition says that, if this elasticity of individual supply of working hours to change in productivity is sufficiently negative, not only endogenous fluctuations, as shown by Iong and Irmen (2021), but also complicated dynamics in the form of odd-period cycles and turbulence will emerge.[16] In other words, if individuals respond to productivity growth by aggressively cutting the working hours, then the long-term evolution of the economy can potentially be complex and unpredictable.[17]

### 3.3. The Matsuyama model: endogenous growth with cycles

Matsuyama (1999) develops an endogenous growth model in which innovation is explicitly introduced with innovators enjoying temporary monopoly rents to study economic growth through cycles. There is no population growth and labor is supplied inelastically. Despite the sharp differences in model setup and the underlying economic mechanism, there is a direct parallelism in the equilibrium dynamics between Matsuyama (1999) and Iong and Irmen (2021).[18] Growth cycles in the Matsuyama model arise from dual forces of economic growth, investment-driven capital accumulation and innovation-driven variety expansion. If capital is scarce relative to the number of varieties, the economy grows purely through capital accumulation with no innovation. If capital is relatively abundant, the research sector becomes active. In equilibrium, the economy fluctuates between a Solow-type neoclassical growth regime and a Romer-type endogenous growth regime. The dynamics of capital stock per variety of intermediate goods $x$ can be described by the M-map

$$f_m(x) = \begin{cases} Gx^{\alpha} & 0 \le x \le 1 \\ \frac{G\beta x}{\beta - 1 + x} & 1 < x < G, \end{cases}$$

[16] One may perhaps also note the emergence of the golden ratio in the Imben-Iong model, a possibility for intertemporal allocation first pointed out by Mitra (1996) and Nishimura and Yano (1996) and further commented on in Khan and Piazza (2011).

[17] We want to remind our reader again of the issue of observability of chaos especially for the interpretation of our results in an applied setting.

[18] From a technical point of view, the equations are very different with the former model having a linear left arm, and the latter with both arms nonlinear.

where both $\alpha$ and $\beta$ depend on the elasticity of substitution between different intermediate inputs satisfying $\alpha \in (0, 1)$ and $\beta = \alpha^{\frac{\alpha}{1-\alpha}} \in (1/e, 1)$ with $\beta$ decreasing with $\alpha$, and a composite parameter $G$ satisfies $1 < G < (1/\beta) - 1$. Let $f_m(G) = f_m^2(1) = \beta G^2/(\beta - 1 + G) \equiv \tau$, with $\beta < 1/2$ (or equivalently, $\alpha > 1/2$). We have $f_m \in \mathcal{F}$ with the modal point $m = 1$, the interior fixed point $z = 1 + \beta G - \beta$ and $\Pi = \{z\}$. If $\tau = f_m^2(m) = f_m^2(1) < 1 = m$, $f_m^3(m) = G\tau^\alpha$. Applying Theorems 2 and 3, we obtain the following conditions for the M-map to admit odd-period cycles and for its second iterate to be turbulent, thus establishing the necessity of those sufficient conditions first identified in Mitra (2001) and Deng and Khan (2018).

**Proposition 4.** *The map $f_m$ has an odd-period cycle if and only if $\tau < 1$ and $G\tau^\alpha < 1 + \beta G - \beta$; $f_m^2$ is turbulent if and only if $\tau < 1$ and $G\tau^\alpha \leq 1 + \beta G - \beta$.*

### 3.4. The Baumol-Wolff model: information production and dissemination

To shed light on the sources of the productivity slowdown in the 1970s, Baumol and Wolff (1992) formalize a dynamic model to investigate the two-way relationship between production and dissemination of information and productivity growth in industry. On the one hand, information output in the research sector raises productivity growth in industry outside the research sector. On the other hand, productivity growth translates into higher price of information which depresses the demand for information output. The feedback mechanism leads to the following law of motion of information output $x$ produced by the research sector, which is of the form of a logistic map

$$f_{bw}(x) = \phi_1 x - \phi_2 x^2$$

where $x$ is in $[0, \phi_1/\phi_2]$ and $\phi_1, \phi_2 > 0$ are composite parameters of the model, both increasing with the demand elasticity of information output and the sensitivity of the price of information to productivity growth outside the research sector. The modal point $m$ is given by $\phi_1/(2\phi_2)$. We further require $\phi_1 \leq 4$ to ensure that $f_{bw}(m) \leq \phi_1/\phi_2$. The map $f_{bw}$ can be viewed as a special case of $f_b$ with $c = 0$ so we apply Proposition 2 to obtain the following corollary.

**Corollary 4.** *Let $\hat{t}$ be the unique real root of the equation $t^3 + t^2 - 5t - 13 = 0$ ($t \approx 2.6786$). Then $f_{bw}$ has an odd-period cycle if and only if $\hat{t} + 1 < \phi_1 \leq 4$ and $f_{bw}^2$ is turbulent if and only if $\hat{t} + 1 \leq \phi_1 \leq 4$.*

The corollary says that for chaos to arise in this feedback model, both the demand elasticity of information output and the price sensitivity of information to productivity growth have to be in some intermediate range. This result is a formal proof of the concluding remarks of the paper that “the intertemporal mechanism may well be oscillatory in character” and that “the feedback process may well be capable of generating chaotic behavior”.

### 3.5. The Deneckere-Judd model: innovation and temporary rents

As a pioneering paper on innovation cycles, Deneckere and Judd (1992) construct a growth model in which innovators enjoy temporary monopoly rents.[19] Unlike Matsuyama (1999), the

[19] For the substantive importance of this model, see Aghion and Howitt (1988, 1992) and Rivera-Batiz and Romer (1991). The “check-map” and the notion of “trapping square” were also to appear first in this paper, concepts which were

model does not incorporate capital accumulation. Entrepreneurs choose to innovate only if the number of varieties of consumption goods is sufficiently small and the innovative activity is profitable. If the number of varieties is sufficiently high, the economy enters a regime with no innovation. With the assumption of exogenous obsolescence of the existing varieties, the number of varieties will gradually decline in the no-innovation regime. In the model, the number of varieties satisfies the following law of motion

$$f_{dj}(x) = \begin{cases} f - \xi x & 0 \le x \le \frac{f}{\xi + (1-d)} \\ \\ (1-d)x & \frac{f}{\xi+(1-d)} < x \le f, \end{cases}$$

where $f > 0$ is a composite parameter, $\xi > 1$ depends on how strong the incentives to innovate decrease with the existing number of varieties, and $0 < d < 1$ captures the degree of variety obsolescence. The map $f_{dj}$ can be viewed as a unimodal map in $\mathcal{F}$ that is turned 180 degree clockwise, and thus it belongs to the class of maps $\mathcal{G}$ as defined at the end of Section 2. Since $f_{dj} \in \mathcal{G}$, we have $m = f/(\xi + 1 - d)$ and $z = f/(\xi + 1)$. Since $\xi > 1$, there is no two-period cycle both periodic points of which are in $[a, m]$ and thus $\Pi' = \{z\}$. We can then apply Corollary 3 to obtain the following conditions for topological chaos, which suggest that for chaos to arise, the incentives to innovate have to strongly decrease with the number of existing varieties and obsolescence of varieties is sufficiently slow.

**Proposition 5.** *The map $f_{dj}$ has an odd-period cycle if and only if $(\xi - 1/\xi)(1-d) > 1$ and $f_{dj}^2$ is turbulent if and only if $(\xi - 1/\xi)(1-d) \ge 1$.*

### 3.6. The Robinson-Solow-Srinivasan model: optimal development policy

Khan and Mitra (2005a, 2012) study the optimal choice of technique in a model of development planning originally formulated by Robinson, Solow, and Srinivasan. In the two-sector Robinson-Solow-Srinivasan model, production technology in both the consumption good and investment good sector is linear. The optimal dynamics of capital stock are investigated through the check-map, as described by

$$f_c(x) = \begin{cases} 1 - \xi x & 0 \le x \le 1/\xi \\ \\ (1-d)x - (1-d)/\xi & 1/\xi \le x \le 1, \end{cases}$$

where $\xi > 1$ is a composite parameter and $0 < d < 1$ is the depreciation rate of capital. It is worth highlighting the analytical parallel between the check-map and the map $f_{dj}$ arising from the Deneckere-Judd model. Since the check-map is also in $\mathcal{G}$, we can apply Corollary 3 to obtain the same conditions for topological chaos as in Proposition 5.

**Corollary 5.** *The map $f_c$ has an odd-period cycle if and only if $(\xi - 1/\xi)(1-d) > 1$ and $f_c^2$ is turbulent if and only if $(\xi - 1/\xi)(1-d) \ge 1$.*

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intensively relied upon in the geometrical conceptions of the work on the Robinson-Solow-Srinivasan model; see Khan and Mitra (2013) and their references.

## 4. Concluding remarks

In this paper, we establish two equivalence results for a class of unimodal maps. The first equivalence result concerns the existence of odd-period cycles, while the second concerns the second iterate of the map to be turbulent. These two equivalence results lead to easily verifiable conditions for topological chaos. Compared with the sufficient condition originally identified in Mitra (2001), our characterization results provide necessary *and* sufficient conditions not only for the turbulence of the second iterate but also for the existence of odd-period cycles. To demonstrate their easy verifiability, we apply our results to six models of economic dynamics and obtain parametric conditions for the two specific forms of topological chaos.

We conclude this paper with two observations, both concerning the new tools for the study of cyclical and chaotic dynamics:

(i) Given its prominent role in complicated dynamics, it is curious whether the existence of three-period cycles can be characterized by easily verifiable necessary and sufficient conditions.
(ii) Recent work suggests the emergence of pan maps with a flat bottom and bimodal maps in the two-sector growth model with linear production technology (Khan and Mitra, 2012; Deng et al., 2021). It remains open whether our characterization results can be extended to incorporate those maps.

So this is the next step in this research program.

## 5. Proofs

**Proof of Lemma 1.** If $f \in \mathcal{F}$ has a periodic cycle of odd period $n \geq 3$, but no periodic cycle of odd period strictly between 1 and $n$, then we can order the $n$ periodic points in increasing order on $[a, b]$ and define the midpoint $c$ for which $\frac{(n-1)}{2}$ periodic points are smaller than $c$ and $\frac{(n-1)}{2}$ periodic points are greater than $c$. We now establish two preliminary results.

**Lemma 2.** *Suppose $f \in \mathcal{F}$ has a periodic cycle of odd period $n > 1$, but no periodic cycle of odd period strictly between 1 and $n$. If $c$ is the midpoint of the cycle of odd period $n$, then the points of the cycle have the order:*

$$f^{n-1}(c) < f^{n-3}(c) < \cdots < f^2(c) < c < f(c) < \cdots < f^{n-2}(c) \tag{1}$$

**Proof.** According to Proposition 8 (p.10) in Block and Coppel (1992), which is originally due to Stefan (1977), we know that the points of the cycle have the order given in (1), or the points of the cycle have the reverse order:

$$f^{n-1}(c) > f^{n-3}(c) > \cdots > f^2(c) > c > f(c) > \cdots > f^{n-2}(c).$$

Using the fact that $f \in \mathcal{F}$, we can now rule out the reverse order as follows. If the reverse order were to hold, then since $c > f(c)$, we must have $c \in (z, b]$. And, since $f^{n-1}(c) > f^{n-3}(c) > \cdots > f^2(c) > c$ holds, we must have $f^{n-1}(c)$, $f^{n-3}(c)$, ..., $f^2(c)$ also in $(z, b]$. Since $f^{n-1}(c) \in (z, b]$, $f^{n-1}(c)$ and $z$ both belong to $[m, b]$, where $f$ is decreasing. Then we must have $f(f^{n-1}(c)) < f(z) = z$. But this means $c = f^n(c) = f(f^{n-1}(c)) < z$, which contradicts the fact that $c \in (z, b]$. Thus, the reverse order cannot hold, and (1) must hold, establishing the lemma. □

It should be noted that a statement like (1) pertains to $n$ odd with $n-1>n-2>n-3>2$; that is, for $n \geq 7$; otherwise, with for example $n=3$, the statement $f^{n-1}(c)>f^2(c)$ is clearly invalid. For $n=3$, the statement of (1) would read with all entries to the left of $f^2(c)$ and all entries to the right of $f(c)$ eliminated: $f^2(c)<c<f(c)$. For $n=5$, the statement of (1) would read with the term $f^{n-3}(c)$ eliminated: $f^4(c)<f^2(c)<c<f(c)<f^3(c)$. Using Lemma 2, we can determine the position of the points of the cycle with respect to $m$ and $z$.

**Lemma 3.** *Suppose $f \in \mathcal{F}$ has a periodic cycle of odd period $n>1$, but no periodic cycle of odd period strictly between 1 and $n$. If $c$ is the midpoint of the cycle of odd period $n$, then the points of the cycle satisfy: (i) $f^{n-1}(c)<m$; (ii) $c<z$; (iii) $f^{n-2}(c) \geq \cdots \geq f(c)>z$; (iv) $c>m$ if $n \geq 5$.*

**Proof.** To establish (i), suppose on the contrary that $f^{n-1}(c) \geq m$. Then since $c>f^{n-1}(c)$, we have both $c$ and $f^{n-1}(c)$ in $[m, b]$, where $f$ is decreasing. Thus, $f(c)<f(f^{n-1}(c))=f^n(c)=c$, which contradicts (1). To establish (ii), note that $f(c)>c$, so that $c<z$. To establish (iii), note that by (1), we have $f^{n-2}(c) \geq \cdots \geq f(c)$. We use the weak inequalities to accommodate the case in which $n=3$. Thus, it remains to establish that $f(c)>z$. Suppose, on the contrary, that $f(c) \leq z$. Then, since $f \in \mathcal{F}$, we must have $f(f(c)) \geq f(c)$, which means $f^2(c) \geq f(c)$, contradicting (1). To establish (iv), suppose on the contrary that $c \leq m$. Since $f^2(c)<c$ by (1), we have both $f^2(c)$ and $c$ in $[a, m]$, where $f$ is increasing. Thus, $f(f^2(c))<f(c)$, which contradicts (1) whenever $n \geq 5$. $\square$

With Lemmas 2 and 3 in hand, we now turn to the proof of Lemma 1. In view of the special status of $n=3$, we first exclude this from our general analysis below. This allows us to display the nature of our proof for $n$ odd and $n \geq 5$. The necessary condition for $n=3$ can be established independently. Let $f \in \mathcal{F}$ have a periodic cycle of odd period $n \geq 5$, but no periodic cycle of odd period strictly between 1 and $n$ and let $c$ be the midpoint of the cycle of odd period $n$. Our proof consists of two main steps.

**Step 1.** We define $x_0=m$, and then proceed to define $(n+1)$ additional points $\{x_1, \ldots, x_{n+1}\}$. By (i) in Lemma 3, we have $f(f^{n-2}(c))=f^{n-1}(c)<m$, while $f(m)>m$. By (iii) in Lemma 3, $f^{n-2}(c)>z$. Since $f^{n-2}(c)>z>m$, there exists $x_1 \in (m, f^{n-2}(c))$ such that $f(x_1)=m=x_0$. Since $f(x_1)=m<x_1$, we have $x_1>z$, and so there exists $x_1 \in (z, f^{n-2}(c))$ such that $f(x_1)=x_0=m$. Define $\mathcal{I}=\{i$ odd, and $i \in\{1, \ldots, n-4\}\}$. We now claim that if for some $i \in \mathcal{I}$, we have $x_i \in (z, f^{n-i-1}(c))$ such that $f(x_i)=x_{i-1}$, then

$$\left.\begin{array}{l}(i)\ \exists x_{i+1} \in (f^{n-i-2}(c), z) \text{ such that } f(x_{i+1})=x_i \\ (ii)\ \exists x_{i+2} \in (z, f^{n-i-3}(c)) \text{ such that } f(x_{i+2})=x_{i+1}\end{array}\right\} \tag{2}$$

To establish (2)(i), note that we have $(n-i-2)$ is even, and so $f^{n-i-2}(c)<c<z$ by (1) and (ii) in Lemma 3. Further, we have $f(f^{n-i-2}(c))=f^{n-i-1}(c)>x_i$, while $f(z)=z<x_i$. Thus, there exists $x_{i+1} \in (f^{n-i-2}(c), z)$ such that $f(x_{i+1})=x_i$, establishing (2)(i). To establish (2)(ii), note that we have $(n-i-3)$ is odd, and so $f^{n-i-3}(c)>z$ by (iii) in Lemma 3. Further, by (2)(i) we have $f(f^{n-i-3}(c))=f^{n-i-2}(c)<x_{i+1}$, while $f(z)=z>x_{i+1}$. Thus, there exists $x_{i+2} \in (z, f^{n-i-3}(c))$ such that $f(x_{i+2})=x_{i+1}$, establishing (2)(ii).

Using the definition of $x_1$ and (2)(i) and (2)(ii), we would then obtain $S=\{x_2, x_3, \ldots, x_{n-3}, x_{n-2}\}$, satisfying $x_{i+1} \in (f^{n-i-2}(c), z)$ such that $f(x_{i+1})=x_i$ and $x_{i+2} \in (z, f^{n-i-3}(c))$ such that $f(x_{i+2})=x_{i+1}$ for $i \in \mathcal{I}$. Together with $x_1$ defined above, we have a set of $(n-2)$ points in the set $\{x_1, \ldots, x_{n-2}\}$. It remains to define three additional points $\{x_{n-1}, x_n, x_{n+1}\}$.

For $i = n-4$, we get $x_{n-2} \in (z, f(c))$ such that $f(x_{n-2}) = x_{n-3}$. Thus, $f(c) > x_{n-2}$, while $f(z) = z < x_{n-2}$. Further, we have $c < z$ by (ii) in Lemma 3, so there exists $x_{n-1} \in (c, z)$ such that $f(x_{n-1}) = x_{n-2}$. Then, we have $f(f^{n-1}(c)) = f^n(c) = c < x_{n-1}$, while $f(m) \geq f(f^{n-3}(c)) = f^{n-2}(c) > z > x_{n-1}$, where the second inequality follows from (iii) in Lemma 3. Further, we have $f^{n-1}(c) < m$ by (i) in Lemma 3, so there exists $x_n \in (f^{n-1}(c), m)$ such that $f(x_n) = x_{n-1}$. Then, we have $f(f^{n-2}(c)) = f^{n-1}(c) < x_n$, while $f(z) = z > m > x_n$. Further, $f^{n-2}(c) > z$ by (iii) in Lemma 3, so there exists $x_{n+1} \in (z, f^{n-2}(c))$ such that $f(x_{n+1}) = x_n$.

**Step 2.** We now use the points $\{x_0, x_1, ..., x_{n+1}\}$ to establish Lemma 1. To this end, note that if $f^j(m) \leq m$ for some odd $j$ with $3 \leq j \leq n$ then we are already done. Thus, we assume that $f^j(m) > m$ for all odd $j$ with $3 \leq j \leq n$. It remains to prove that $f^{n+2}(m) \leq m$.

Since $x_{n+1} \in (z, f^{n-2}(c))$, we have $f(m) \geq f(f^{n-3}(c)) = f^{n-2}(c) > x_{n+1} > z$. Thus, $f(m)$ and $x_{n+1}$ belong to $[m, b]$, where $f$ is decreasing, and so $f^2(m) < f(x_{n+1}) = x_n$. Further, we have $x_n < m$. Thus, both $f^2(m)$ and $x_n$ belong to $[a, m]$, where $f$ is increasing, and so $f^3(m) < f(x_n) = x_{n-1}$. Define $J = \{j \text{ odd: } j \in \{3, ..., n\}\}$. We now claim that if $f^j(m) < x_{n-j+2}$ for some $j \in J$, then:

$$\left.\begin{array}{l} (i)\ f^{j+1}(m) > f(x_{n-j+2}) = x_{n-j+1} \\ (ii)\ f^{j+2}(m) < f(x_{n-j+1}) = x_{n-j} \end{array}\right\} \tag{3}$$

To establish (3)(i), note that $x_{n-j+2} > f^j(m) > m$, where the second inequality follows from the assumption that $f^j(m) > m$ for all odd $j$ with $3 \leq j \leq n$. Thus, $x_{n-j+2}$ and $f^j(m)$ belong to $[m, b]$ where $f$ is decreasing, so that $f^{j+1}(m) > f(x_{n-j+2}) = x_{n-j+1}$, establishing (3)(i). For $j \in J$, we have $(n-j+2)$ is even and $n-j+2 \in \{2, ..., n-1\}$; thus, $(n-j+1)$ is odd and $n-j+1 \in \{1, ..., n-2\}$. That is, for $j \in J$, we have $x_{n-j+1} \in \{x_1, x_3, ..., x_{n-2}\}$. Now, note that by construction, all elements of the set $\{x_1, x_3, ..., x_{n-2}\}$ exceed $z > m$. Thus, for $j \in J$, by using (3)(i), we have $f^{j+1}(m) > f(x_{n-j+2}) = x_{n-j+1} > m$, and so both $f^{j+1}(m)$ and $x_{n-j+1}$ belong to $[m, b]$, where $f$ is decreasing. Thus, using (3)(i), we obtain $f^{j+2}(m) < f(x_{n-j+1}) = x_{n-j}$, establishing (3)(ii).

The implication of (3) is that if $f^j(m) < x_{n-j+2}$ for some $j \in J$, with $j \leq n-2$, then $f^{j+2}(m) < f(x_{n-j+1}) = x_{n-(j+2)+2}$, and since $(j+2) \in J$, (3) can be applied again. Since we have shown that $f^j(m) < x_{n-j+2}$ for $j = 3 \leq n-2$, we can use this implication to obtain $f^j(m) < x_{n-j+2}$ for all $j$ odd in $\{3, ..., n\}$. Thus, in particular, we have $f^n(m) < x_2$. Since $n \in J$, one can apply (3) one more time to obtain $f^{n+2}(m) < f(x_1) = x_0 = m$, which is the desired result.

We now turn to the proof of the necessary condition for $n = 3$. Let $c$ be the midpoint of the cycle of period three. Then by Lemmas 2 and 3, we have $f^2(c) < c < f(c)$ and (i) $f^2(c) < m$; (ii) $c < z$; (iii) $f(c) > z$. The proof strategy is very similar to the proof for the general case, which again consists of two main steps.

**Step 1.** We define $x_0 = m$, and then define $\{x_1, x_2, x_3, x_4\}$ as follows. We have $f(c) > z > m$ by (iii) and $f(f(c)) = f^2(c) < m$ by (i), while $f(m) > m$ (since $f \in \mathcal{F}$). Thus, there exists $x_1 \in (m, f(c))$ such that $f(x_1) = m = x_0$. Since $f(x_1) = m < x_1$, we must have $x_1 > z$. So, there exists $x_1 \in (z, f(c))$ such that $f(x_1) = m = x_0$. Then, we have $f(c) > x_1$, while $f(z) = z < x_1$. Further, by (ii), we have $c < z$, so there exists $x_2 \in (c, z)$ such that $f(x_2) = x_1$. Next, note that $f(f^2(c)) = f^3(c) = c < x_2$, while $f(m) \geq f(c) > z$ by (iii) and $z > x_2$. Further, we have $f^2(c) < m$ by (i), so there exists $x_3 \in (f^2(c), m)$ such that $f(x_3) = x_2$. Finally, note that $f(f(c)) = f^2(c) < x_3$, while $f(z) = z > m > x_3$. Further, we have $f(c) > z$ by (iii), so there exists $x_4 \in (z, f(c))$ such that $f(x_4) = x_3$.

**Step 2.** We now use the points $\{x_1, ..., x_4\}$ to establish Lemma 1. To this end, note that if $f^3(m) \leq m$, then we are already done. Thus, it remains to show that if $f^3(m) > m$, then $f^5(m) \leq m$. We start by noting that $f(m) \geq f(c) > x_4 > z$. Thus, $f(m)$ and $x_4$ are in $[m, b]$, where $f$ is decreasing, so $f^2(m) < f(x_4) = x_3$. Next, note that $x_3 < m$, so $f^2(m)$ and $x_3$ are in $[a, m]$, where $f$ is increasing, so $f^3(m) < f(x_3) = x_2$. Recall that we are given $f^3(m) > m$, so we have $f^3(m)$ and $x_2$ in $[m, b]$, where $f$ is decreasing, so $f^4(m) > f(x_2) = x_1$. Since by construction $x_1 > z > m$, so we have $f^4(m)$ and $x_1$ in $[m, b]$, where $f$ is decreasing, so $f^5(m) < f(x_1) = x_0 = m$, which is the desired result. □

**Proof of Theorem 1.** We first prove the "if" part. If the condition holds for some odd $k > 1$, then we have $f^k(m) \leq m < f(m)$ where the second inequality follows from $f \in \mathcal{F}$. Then, by Theorem A of Block and Coppel (1986), which itself is obtained from Li et al. (1982), $f$ has a $k$-period cycle.

We now turn to the "only if" part. If $f$ admits an odd-period cycle, then there is some $n \in \mathbb{N}$, with $n$ odd and $n > 1$, such that $f$ has an $n$-period cycle. Then, we claim that $f^j(m) \leq m$ for some odd $j$ with $3 \leq j \leq n+2$. To see this, since $f \in \mathcal{F}$ has a periodic cycle of odd period $n > 1$, the set $O(f) = \{i \in \mathbb{N} : i \text{ is odd and } i > 1, \text{and } f \text{ has an } i\text{-period cycle}\}$ is non-empty. Since $O(f)$ is a subset of the set of natural numbers, it has a minimum. Denote this minimum by $n'$, and note that $n \geq n' \geq 3$. If $n' = 3$, then by applying Lemma 1, $f^j(m) \leq m$ for some $j \in \{3, 5\}$, so that $f^j(m) \leq m$ for some odd $j$ with $3 \leq j \leq n'+2 \leq n+2$, thus establishing the claim. If $n' \neq 3$, then $n' \geq 5$, and, since $n'$ is the minimum of $O(f)$, there is no periodic cycle of odd period strictly between 1 and $n'$. Then by Lemma 1, we have $f^j(m) \leq m$ for some odd $j$ with $3 \leq j \leq n'+2 \leq n+2$, again establishing the claim. Since $f^j(m) \leq m$ for some odd $j > 1$, we have obtained the desired result. □

**Proof of Theorem 2.** We first establish the following preliminary result.

**Lemma 4.** *Let $f \in \mathcal{F}$. If $f^k(m) \leq m$ for some odd $k > 1$, then $f^2(m) < m$ and $f^3(m) < z$.*

**Proof.** We first prove $f^2(m) < m$. Suppose on the contrary $f^2(m) \geq m$. Since $f(m) > z > m$, $f^2(m) = f(f(m)) < f(z) = z$. Then we must have $z > f^2(m) \geq m$. We now claim that if $z > f^n(m) \geq m$ for some even number $n$, then (i) $z < f^{n+1}(m) \leq f(m)$ and (ii) $z > f^{n+2}(m) \geq m$. Since $z$, $f^n(m)$, and $m$ are in $[m, b]$ where $f$ is decreasing, $z < f^{n+1}(m) \leq f(m)$, thus establishing (i). Since again $z$, $f^{n+1}(m)$ and $f(m)$ are in $[m, b]$, and $f^2(m) \geq m$, $z > f^{n+2}(m) \geq f^2(m) \geq m$, thus establishing (ii). Since $z > f^2(m) \geq m$, using (i) and (ii), we obtain by induction $f^k(m) > z > m$ for any odd $k$, which leads to a contradiction. Thus, we must have $f^2(m) < m$.

We now turn to the second condition, $f^3(m) < z$. Pick the smallest odd $k$ such that $f^k(m) \leq m$. If $k = 3$, it immediately follows from $f^3(m) \leq m < z$. We now claim that, for odd $k$ with $k > 3$, if $f(m) > m$, $f^3(m) > m$,..., $f^{k-2}(m) > m$, and $f^k(m) \leq m$, then $f^3(m) < z$. Suppose on the contrary that $f^3(m) \geq z$. Since $f^3(m) \geq z$, we must have $f^4(m) = f(f^3(m)) \leq f(z) = z$. We claim that if $f^\ell(m) \leq z$ for $\ell \in \{i \text{ even, and } 4 \leq i \leq k-1\}$, then (iii) $f^{\ell+1}(m) \geq z$ and (iv) $f^{\ell+2}(m) \leq z$. Since we have shown $f^2(m) < m$, for any $x$ in $[f^2(m), f(m)]$, $f^2(m) \leq f(x) \leq f(m)$. Since $f^2(m) \in [f^2(m), f(m)]$, by induction we have $f^n(m) \in [f^2(m), f(m)]$ for any natural number $n$ such that $n \geq 2$. In particular, $f^\ell(m) \geq f^2(m)$. Since $f^\ell(m) \leq z$, consider two possible cases: (a) $f^\ell(m) \leq m$, or (b) $m < f^\ell(m) \leq z$. For (a), since $m \geq f^\ell(m) \geq f^2(m)$, $f^{\ell+1}(m) = f(f^\ell(m)) \geq f(f^2(m)) = f^3(m) \geq z$, and then $f^{\ell+2}(m) \leq z$. For (b), since

$f$ is decreasing on $[m, b]$ and $m < f^\ell(m) \le z$, $f(f^\ell(m)) = f^{\ell+1}(m) \ge z$. Then we must have $f^{\ell+2}(m) \le z$. Thus we establish (iii) and (iv). Since $f^4(m) \le z$, using (iii) and (iv), we obtain by induction $f^k(m) \ge z > m$, contradicting to $f^k(m) \le m$. Thus, we must have $f^3(m) < z$, which is the desired result. □

With this lemma in hand, we are now ready to prove the theorem. According to Theorem 1, it suffices to show that Condition OPC is satisfied if and only if $f^k(m) \le m$ for some odd $k > 1$.

We first prove the "if" part. According to Lemma 4, we have $f^2(m) < m$ and $f^3(m) < z$. If $\Pi$ is a singleton, then we reach the desired conclusion. Suppose $\Pi$ is not a singleton. We must have $\underline{\mu} = \min\{x \in \Pi\} < z$. To see this, pick $x \in \Pi$ with $x \neq z$. If $x < z$, then $\underline{\mu} = \min\{x \in \Pi\} \le x < z$. If $x > z$, $f(x) < z$. Since $x \in \Pi$, $f(x) \ge m$ and we know $f^2(f(x)) = f(f^2(x)) = f(x)$, so $f(x) \in \Pi$, which again implies $\underline{\mu} \le f(x) < z$. By construction, $\underline{\mu} \ge m$, and since $f^2(m) < m$, $\underline{\mu} \neq m$. Thus, $\underline{\mu} > m$. Suppose on the contrary $f^3(m) \ge \underline{\mu}$. Since $z > f^3(m)$, we have $z > f^3(m) \ge \underline{\mu}$. We claim if $z > f^k(m) \ge \underline{\mu}$ for odd $k > 1$, then $z > f^{k+2}(m) \ge \underline{\mu}$. Since $\underline{\mu} > m$ and $z > f^k(m) \ge \underline{\mu}$, we have $z > f^k(m) \ge \underline{\mu} > m$, which implies $z = f(z) < f(f^k(m)) = f^{k+1}(m) \le f(\underline{\mu})$. Since $f(\underline{\mu}) \ge f^{k+1}(m) > z > m$, we have $z = f(z) > f(f^{k+1}(m)) = f^{k+2}(m) \ge f(f(\underline{\mu})) = \underline{\mu}$, establishing the claim. Since $z > f^3(m) \ge \underline{\mu}$, by induction, we have $f^k(m) \ge \underline{\mu}$ for any odd $k > 1$. Since $\underline{\mu} > m$, then $f^k(m) > m$ for any odd $k > 1$, which contradicts to the condition that $f^k(m) \le m$ for some odd $k > 1$. Therefore, we must have $f^3(m) < \underline{\mu}$.

We now turn to the "only if" part. Condition OPC is satisfied and suppose on the contrary $f^k(m) > m$ for any odd $k > 1$. To proceed, we define a mapping $f_r : [m, f(m)] \to [f^2(m), f(m)]$ by setting $f_r(x) = f(x)$ for any $x \in [m, f(m)]$. Since $f_r$ is strictly decreasing, it is one to one and so $f_r^{-1}$ is well-defined. We define another mapping $g : [f^2(m), z] \to [m, z]$ by setting $g(x) = f_r^{-1} \circ f_r^{-1}(x)$ for any $x \in [f^2(m), z]$. By construction, $g$ is strictly increasing in $[f^2(m), z]$ with $g(f^2(m)) = m$ and $\underline{\mu}$ is by definition the smallest fixed point of $g$. Since $f^2(m) < m$, $\underline{\mu} \neq m$ and so $\underline{\mu} > m$.

We claim for $n \in \mathbb{N}$ and $x \in [m, \underline{\mu})$, if $f^{2n+3}(m) > x$ and $f^{2n+1}(m) > m$, then

$$f^{2n+1}(m) > g(x). \tag{4}$$

Suppose on the contrary $f^{2n+1}(m) \le g(x)$. Since $f^{2n+1}(m) > m$, $f^{2n+2}(m) = f(f^{2n+1}(m)) \ge f(g(x))$. Since $g(\underline{\mu}) = \underline{\mu}$ and $g$ is strictly increasing, we must have $g(x) < g(\underline{\mu}) = \underline{\mu}$ for any $x$ in $[m, \underline{\mu})$. Also we know $g(x) \ge m$, so $f(g(x)) > f(\underline{\mu}) \ge m$. Hence, $f^{2n+2}(m) \ge f(g(x)) > m$, which implies $f^{2n+3}(m) = f(f^{2n+2}(m)) \le f(f(g(x))) = f_r^2(g(x)) = x$, contradicting to $f^{2n+3}(m) > x$. Therefore, (4) must hold.

Since $g(x) \in [m, z]$ for any $x \in [f^2(m), z]$ and $f^2(m) < m < z$, $z \ge g(m) \ge m$, and by $f^2(m) < m$, $g(m) \neq m$, so $z \ge g(m) > m$. Since $g$ is strictly increasing, $z = g(z) \ge g(g(m)) = g^2(m) > g(m)$. By induction, $z = g(z) \ge g(g^n(m)) = g^{n+1}(m) > g(g^{n-1}(m)) = g^n(m) > m$ for any natural number $n$. So the sequence $\{g^n(m)\}_{n=0}^\infty$ is well defined and it is strictly increasing. Since the sequence has an upper bound $z$, there must exist a limit. Suppose $\lim_{n\to\infty} g^n(m) < \underline{\mu}$. Since $g$ is continuous, $g(\lim_{n\to\infty} g^n(m)) = \lim_{n\to\infty} g^{n+1}(m) = \lim_{n\to\infty} g^n(m)$, which contradicts to $\underline{\mu}$ being the smallest fixed point of $g$. Therefore, $\lim_{n\to\infty} g^n(m) \ge \underline{\mu}$. Further, we know $f^3(m) < \underline{\mu}$ and by supposition $f^3(m) > m$. Since the sequence $\{g^n(m)\}_{n=0}^\infty$ is strictly increasing, $\lim_{n\to\infty} g^n(m) \ge \underline{\mu}$, and $\underline{\mu} > m$, there exists a natural number $n_0$ such that $g^n(m) > f^3(m)$

for any $n \geq n_0$ and $g^n(m) \leq f^3(m)$ for any non-negative integer $n$ such that $n < n_0$. Since $f^k(m) > m$ for any odd $k > 1$, in particular, $f^{2n_0+3}(m) > m$ and $f^{2n_0+1}(m) > m$. Then, by (4), $f^{2n_0+1}(m) > g(m)$. Since $g^n(m) \leq f^3(m) < \underline{\mu}$ for $n = 0, 1, 2, ..., n_0 - 1$, applying the same argument using (4), we obtain $f^{2n_0-2n+3}(m) > g^n(m)$ for $n = 1, 2, ..., n_0$. In particular, $f^3(m) > g^{n_0}(m)$, which leads to a contradiction. So there must exist some odd $k > 1$ such that $f^k(m) \leq m$.

Hence, we have shown that Condition OPC is satisfied if and only if $f^k(m) \leq m$ for some odd $k > 1$. The necessity and sufficiency of Condition OPC for the odd-period cycles then directly follows from Theorem 1. $\square$

**Proof of Theorem 3.** We first prove the "if" part. As a preliminary step, let us note that since $f^2(m) < m$, the modal point $m$ does not belong to $\Pi$. That is, $\Pi$ is actually a subset of $(m, b]$. Let $\pi \equiv \bar{\mu} = \max\{x \in \Pi\}$, and define $\pi' = f(\pi)$. Then, by the definition of $\Pi$, we have $\pi' \in D = [m, b]$. Further, $f(\pi') = f(f(\pi)) = \pi$, and since $\pi \in \Pi \subset D$, we have $f(\pi') \in D$. Finally, $f^2(\pi') = f(\pi) = \pi'$. Thus $\pi'$ also belongs to the set $\Pi$. In particular, let us note for what follows that

$$f(\pi') = \pi \in (m, b] \text{ and } \quad \pi' = f(\pi) \in (m, b] \tag{5}$$

We divide the proof now into two cases: (i) $f^2(m) < m$ and $f^3(m) = \pi$; (ii) $f^2(m) < m$ and $f^3(m) < \pi$.

**Case (i):** In this case, we construct a trajectory from $m$, such that $f^2$ reaches $\pi' = f(\pi)$ in two periods ($f$ reaches $\pi'$ in four periods) after moving away from $\pi'$ in the first period. Define $p = f^2(m)$. Then, by (5), $\pi' > m$, and since $f^2(m) < m$, we have $p < m$. We note this as $p < m < \pi'$. Since $f^3(m) = \pi$, we have $f(p) = f(f^2(m)) = f^3(m) = \pi$, and so by (5): $f^2(p) = f(f(p)) = f(\pi) = \pi'$. We obtain $f^2(p) = \pi' = f^2(\pi')$, where the second equality follows from the definition of $\pi'$, and $p = f^2(m)$. Since we know $p < m < \pi'$, we conclude that $f^2$ is turbulent.

**Case (ii):** In this case, we construct a trajectory from a point *exceeding* $m$ (point identified as $s'$ below) such that $f^2$ reaches $\pi$ in two periods ($f$ reaches $\pi$ in four periods) after moving away from $\pi$ in the first period. This construction is naturally more involved than in case (i) above. It makes repeated use of the intermediate value theorem, and the trajectory is defined by "backward recursion" from $\pi$.

By (5), $\pi' > m$, and since $\pi', m \in D$, we have $f(m) > f(\pi') = \pi$. Since $f^3(m) < \pi$, we have $f(f^2(m)) < \pi$. Using the continuity of $f$ and noting that $f^2(m) < m$, there is $p' \in (f^2(m), m)$ such that $f(p') = \pi$. By (5) and since $p' < m$, we have $f(\pi) = \pi' > m > p'$. Further, $f(f(m)) = f^2(m) < p'$. Using the continuity of $f$ and noting that $f(m) > \pi$, there is:

$$q' \in (\pi, f(m)) \text{ such that } \quad f(q') = p' \tag{6}$$

We have, by (5), $f(\pi) = \pi' > m$. Also, since $f^2(m) < m$, we have $f(f(m)) = f^2(m) < m$. Using the continuity of $f$ and noting that $f(m) > \pi$, there is:

$$r' \in (\pi, f(m)) \text{ such that } \quad f(r') = m \tag{7}$$

Note that both $q'$ and $r'$ belong to $(\pi, f(m)) \subset D$. Since we have $p' < m$ (by the construction of $p'$), we obtain $f(q') = p' < m = f(r')$. Since $f$ is decreasing on $D$, and $q'$ and $r'$ belong to $D$, implies that $q' > r'$. We have from (6) and (7), $f^2(r') = f(m) > q'$. Also, since $\pi \in \Pi$, we can

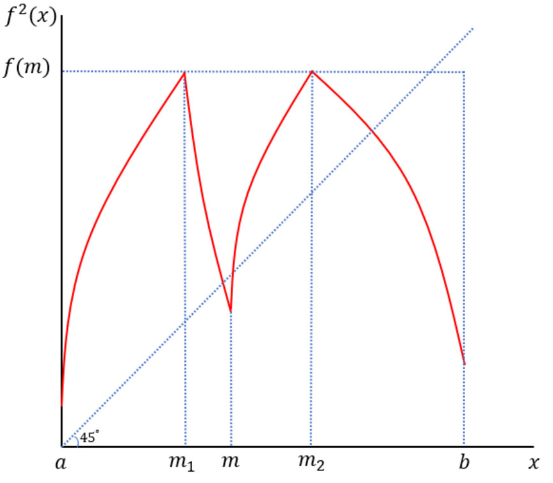


Fig. 2. The Second Iterate of $f$.

use (6) to write $f^2(\pi) = \pi < q'$. Using the continuity of $f^2$ and noting that $\pi < r'$ (from (7)), there is:

$$s' \in (\pi, r') \text{ such that } \quad f^2(s') = q' \tag{8}$$

We now summarize the information from the previous steps. Note that $q' > r'$ and (8) that $\pi < s' < r' < q'$. Further, using $p' < m$, (6), (8), and the fact that $\pi \in \Pi$, we obtain $f^2(q') = f(p') = \pi = f^2(\pi)$ and $f^2(s') = q'$. Thus, we conclude that $f^2$ is turbulent.

We now turn to the "only if" part. It is helpful to get a visual of the second iterate of $f \in F$. It is useful for this purpose to assume that $f(a) < m$ and $f(b) < m$. This assumption is a simplification, allowing us to draw a diagram rigorously with the maintained assumptions. As will be clear later, it can be dispensed with for the theory to follow. Since $f(a) < m$ and $f(m) > m$, there is $m_1 \in (a, m)$ such that $f(m_1) = m$. Since $f(m) > m$ and $f(b) < m$, there is $m_2 \in (m, b)$ such that $f(m_2) = m$. With this information in hand, we have $a < m_1 < m < m_2 < b$ and the second iterate of $f$ (that is, $f^2$) has three "turning points" $m_1, m$, and $m_2$ and four sections $S_1 = [a, m_1]$, $S_2 = [m_1, m]$, $S_3 = [m, m_2]$ and $S_4 = [m_2, b]$, such that $f^2$ is monotone increasing on $S_1$, monotone decreasing on $S_2$, monotone increasing on $S_3$ and monotone decreasing on $S_4$.

This information allows us to draw a diagram of $f^2$ as Fig. 2. It is meant to be a visual aid to our analysis. There is one observation about the diagram that is worth noting. Anticipating the theory to come, pertaining to the implications of turbulence of $f^2$, we have assumed in the diagram that $f^2(m) < m$. Other than this feature, the diagram is meant to be generic for maps $f \in \mathcal{F}$ satisfying $f(a) < m$ and $f(b) < m$.

If $f^2$ is turbulent, then there exist points $x_1, x_2, x_3$ in $[a, b]$, satisfying:

$$f^2(x_2) = f^2(x_1) = x_1 \text{ and } \quad f^2(x_3) = x_2 \tag{9}$$

and, in addition, either:

$$\left.\begin{array}{ll} (i)\ x_1 < x_3 < x_2 & \\ (ii)\ f^2(x) > x_1 & \text{for } x_1 < x < x_2 \\ (iii)\ x < f^2(x) < x_2 & \text{for } x_1 < x < x_3 \end{array}\right\} \tag{10}$$

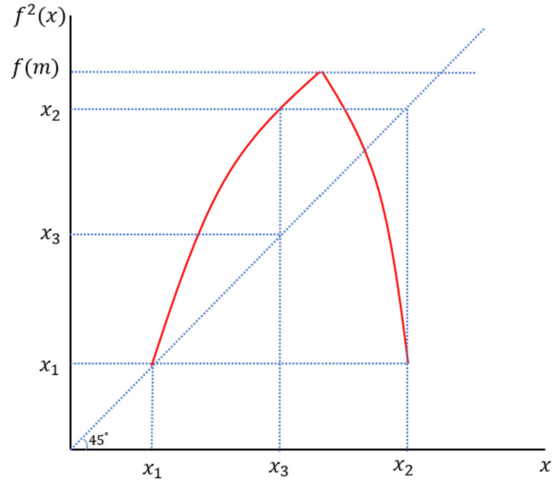


Fig. 3. Implication of Turbulence for $f^2$.

or the same with all inequalities in (10) reversed. This follows from Lemma 1 in Block and Coppel (1986). Note well the additional conditions (10)(ii) and (10)(iii), not mentioned in the sufficient conditions for turbulence of $f^2$ in Block and Coppel (1986). The information above allows us to draw Fig. 3, which reflects the nature of the map $f^2$, on the interval $[x_1, x_2]$, when $f^2$ is turbulent. In drawing this diagram, we have made use of the information in (9) and (10), but also the information that the map of $f^2$ is piecewise monotone.

We will in fact suppose that (10) is satisfied, since the other case is entirely analogous. We proceed with our analysis by separating two cases (i) $x_1 \geq m$, (ii) $x_1 < m$. Visually, the first case can be identified using Figs. 2 and 3 as one that arises when sections $S_3$ and $S_4$ contain the interval $[x_1, x_2]$. We investigate case (i) first.

**Case (i):** Since $f$ is decreasing on $[m, b]$, $f$ decreases from $f(m)$ to $f(b)$ as $x$ increases from $m$ to $b$. We now claim that $f(b) < m$. Suppose, on the contrary, that $f(b) \geq m$. Then $f(m) \geq f(b) \geq m$, so that both $f(m)$ and $f(b)$ belong to $[m, b]$, and since $f$ is decreasing on $[m, b]$, we must have $f^2(x)$ increasing from $f^2(m)$ to $f^2(b)$ as $x$ increases from $m$ to $b$. Since (10)(i) gives us $x_2 > x_3 > x_1 \geq m$, we must therefore have $f^2(x_2) > f^2(x_3)$. But this implies by (9) that $x_1 > x_2$, a contradiction. This establishes the claim.

Since $f(m) > m$ and $f(b) < m$, there is $m_2 \in (m, b)$ such that $f(m_2) = m$. With this information in hand, we have sections $S_3 = [m, m_2]$ and $S_4 = [m_2, b]$, such that $f(x)$ decreases from $f(m)$ to $f(b)$ as $x$ increases from $m$ to $b$. Since $f(m_2) = m$, and $f$ is decreasing on $[m, b]$ and increasing on $[a, m]$, we must have $f^2$ monotone increasing on $S_3$ and monotone decreasing on $S_4$. Note that by (9) and (10)(ii), $x_1$ is a fixed point of $f^2$ in an upward sloping section of $f^2$. Since $x_1 \geq m$, we must therefore have $x_1$ in $S_3 = [m, m_2]$. We now proceed to establish properties of $x_3$, $x_2$, and $x_1$ (in that order), which will in turn lead to our desired conclusion that Condition TUR holds. Specifically, we proceed to establish:

$$(i)\; x_3 \in S_3; \quad (ii)\; x_2 > m_2; \quad (iii)\; f^2(m) < m; \quad (iv)\; x_1 \in \Pi. \tag{11}$$

To establish (11)(i), suppose on the contrary that $x_3 \notin S_3$. Then, since $x_3 > x_1 \geq m$, we must have $x_3 \in (m_2, b]$, and so $m_2 \in [x_1, x_3)$. That is either $m_2 = x_1$ or $m_2 \in (x_1, x_3)$. In the first case, $f^2(m_2) = f^2(x_1) = x_1 < x_2 = f(f(x_3)) \leq f(m)$, a contradiction to the definition of $m_2$. In the second case, $f^2(m_2) = f(m) \geq f(f(x_3)) = x_2$, which contradicts (10)(iii). This establishes (11)(i), and we have $m \leq x_1 < x_3 \leq m_2$. Note that it is possible that $x_3 = m_2$, or that $x_3 < m_2$. We have drawn the latter case in Fig. 3, since the former might be viewed as coincidental.

To establish (11)(ii), suppose again on the contrary that $x_2 \leq m_2$. Then, since $x_2 > x_1 \geq m$, we must have $x_2 \in S_3$. Since $x_3$ also belongs to $S_3$ by (11)(i), with $x_2 > x_3$, and $f^2$ is increasing on $S_3$, we have $x_1 = f^2(x_2) > f^2(x_3) = x_2$, a contradiction to (10)(i). This establishes (11)(ii).

To establish (11)(iii), suppose on the contrary that $f^2(m) \geq m$. Then, we have $f(f(m)) \geq m = f(m_2)$. Since $f(m)$ and $m_2$ belong to $[m, b]$, where $f$ is decreasing, we must therefore have $f(m) \leq m_2$. Using (11)(ii), we then obtain $f(m) \leq m_2 < x_2 = f^2(x_3) = f(f(x_3))$, which contradicts the fact that $f$ is unimodal on $[a, b]$, with modal point $m$. This establishes (11)(iii), which is the first part of Condition TUR.

To establish (11)(iv), recall that $x_1$ is a fixed point of $f^2$ in $S_3$. Since $f^2(m) < m$, $x_1$ cannot be equal to $m$. Also, since $x_1 < x_3 \leq m_2 \leq b$, we have $x_1 \in (m, m_2) \subset (m, b)$. Define $x_4 = f(x_1)$. Note that $f(x_4) = f^2(x_1) = x_1$, and so $f^2(x_4) = f(x_1) = x_4$. Thus, $x_4$ is also a fixed point of $f^2$. To show that $x_1 \in \Pi$, it remains to verify that $x_4 \in [m, b]$. To prove that $x_4 \in [m, b]$, note that $f(x_4) = x_1 \in (m, m_2)$, so that $m < f(x_4) < m_2$; then, since $f$ is decreasing on $[m, b]$, we must have $x_4 = f^2(x_4) > f(m_2) = m$. This establishes (11)(iv).

We now use (11) to establish that $f^3(m) \leq x_1$. We have $f(m) \geq f(f(x_3)) = x_2$ using the fact that $f$ is unimodal on $[a, b]$ with modal point $m$, and (9). Since $x_2 > m_2$ by (11)(ii) and $f(m) \geq x_2$, $f(m) > m_2$. Since $f^2$ is decreasing on $S_4$ and $f(m) \geq x_2$, we have $f^3(m) = f^2(f(m)) \leq f^2(x_2) = x_1$. Since $x_1 \in \Pi$ by (11)(iv), this implies $f^3(m) \leq \bar{\mu}$, thus establishing the second part of Condition TUR.

**Case (ii):** We separate the analysis in this case, where $x_1 < m$, to three subcases: (I) $a < f(a) < m$; (II) $f(a) \geq m$; (III) $f(a) = a$. We show that, in fact, subcases I and II can be ruled out, and the only subcase that is possible (when $x_1 < m$) is subcase III.

Subcase I: Since $f(a) < m$ and $f(m) > m$, there is $m_1 \in (a, m)$ such that $f(m_1) = m$. With this information in hand, we have $a < m_1 < m$ and there are sections $S_1 = [a, m_1]$, $S_2 = [m_1, m]$, such that $f(x)$ increases from $f(a)$ to $f(m_1)$ as $x$ increases from $a$ to $m_1$, and $f(x)$ increases from $f(m_1) = m$ to $f(m) > m$ as $x$ increases from $m_1$ to $m$. Since $f$ is increasing on $[a, m]$ and decreasing on $[m, b]$, we must have $f^2$ monotone increasing on $S_1$ and monotone decreasing on $S_2$. Further, note that for $x \in [a, m_1]$, we have $x < f(x) \in [f(a), m]$, so that $f^2(x) > f(x) > x$ for all $x \in [a, m_1]$. That is, $f^2$ has no fixed point in $[a, m_1]$. Since $x_1$ is a fixed point of $f^2$ in an upward sloping portion of the map (by (9) and (10)(ii)), $x_1 < m$ is not possible. That is, in Case (ii), where $x_1 < m$, Subcase I cannot arise.

Subcase II: Note that $f(x)$ is increasing from $f(a)$ to $f(m)$ as $x$ increases from $a$ to $m$. Since $f(a) \geq m$, and $f$ is decreasing on $[m, b]$, we must have $f^2(x)$ decreasing from $f^2(a)$ to $f^2(m)$. Since $x_1$ is a fixed point of $f^2$ in an upward sloping portion of the map (by (9) and (10)(ii)), $x_1 < m$ is not possible. That is, in Case (ii), where $x_1 < m$, Subcase II cannot arise.

Subcase III: Since $f(a) = a < m$ and $f(m) > m$, there is $m_1 \in (a, m)$ such that $f(m_1) = m$. With this information in hand, we have $a < m_1 < m$ and there are sections $S_1 = [a, m_1]$, $S_2 = [m_1, m]$, such that $f(x)$ increases from $f(a)$ to $f(m_1)$ as $x$ increases from $a$ to $m_1$, and $f(x)$ increases from $f(m_1) = m$ to $f(m) > m$ as $x$ increases from $m_1$ to $m$. Since $f$ is increasing on $[a, m]$ and decreasing on $[m, b]$, we must have $f^2$ monotone increasing on $S_1$ and monotone decreasing on $S_2$. Further, note that for $x \in (a, m_1]$, we have $x < f(x) \in (f(a), m]$, so that $f^2(x) > f(x) > x$ for all $x \in (a, m_1]$. That is, $f^2$ has no fixed point in $(a, m_1]$. Since $x_1$ is a fixed point of $f^2$ in an upward sloping portion of the map (by (9) and (10)(ii)), and $x_1 < m$, we must have $x_1 = a$.

Note that we have $f(f(x_2)) = x_1 = a$. Since $f(x) > a$ for all $x \in (a,b)$, we must have either (a) $f(x_2) = a = x_1$, or (b) $f(x_2) = b$. In Case (a), note again that since $f(x) > a$ for all $x \in (a,b)$, we must have $x_2 = a$ or $x_2 = b$. And since $x_2 > x_1 = a$, we must in fact have $x_2 = b$, and consequently $f(b) = a$. In Case (b), we have $a = x_1 = f^2(x_2) = f(b)$, so again we must have $f(b) = a$. Thus, in both Case (a) and Case (b), we have $f(b) = a$. In both Case (a) and Case (b), we must also have $f(m) = b$. To see this, note that in Case (a), we have $b \geq f(m) \geq f(f(x_3)) = f^2(x_3) = x_2 = b$, while in Case (b), we have $b \geq f(m) \geq f(x_2) = b$. Since $f(m) = b$ and $f(b) = a$, $f^2(m) = f(b) = a < m$ and $f^3(m) = f(a) = a \leq z \leq \bar{\mu} = \max\{x \in \Pi\}$. Clearly, these two conditions verify Condition TUR.

Thus, $f^2$ is turbulent if and only if Condition TUR is satisfied. $\square$

**Proof of Proposition 1.** By the definition of turbulence, there exists three points $x_1$, $x_2$, and $x_3$, in $[a,b]$ such that $f(x_2) = f(x_1) = x_1$ and $f(x_3) = x_2$ with either (1) $x_1 < x_3 < x_2$ or (2) $x_2 < x_3 < x_1$. Note that $x_1$ is a fixed point of $f$ and we first claim that $x_1$ is not the unique interior fixed point $z$. Suppose on the contrary $x_1 = z$. If (1) holds, then $x_3 > x_1 = z$ and therefore $x_2 = f(x_3) < z = x_1$, contradicting to $x_2 > x_1$. If (2) holds, then $x_3 < x_1 = z$. We know $x_2 = f(x_3)$ and $x_2 < x_3$, so $x_3 > m$. Since $x_1 > x_3 > m$, $x_2 = f(x_3) > f(x_1) = x_1$, contradicting to $x_2 < x_1$. This establishes the claim that $x_1 \neq z$.

Since $x_1$ is a fixed point and $x_1 \neq z$, we must have $a = x_1$ being a fixed point. This also rules out (2). Since $x_2 > x_1 = a$ and $f(x) > x$ for all $x \in (a,z)$, we must have $x_2 \in (z,b]$. If $x_2 < b$, then we would have $f(b) < f(x_2) = a$, contradicting to the fact that $f(x) \geq a$ for any $x \in [a,b]$. Thus, we must have $x_2 = b$ and $f(b) = f(x_2) = x_1 = a$. Finally, we know $b \geq f(m) \geq f(x_3) = x_2 = b$, so we must have $x_3 = m$ and $f(m) = b$. $\square$

**Proof of Corollary 3.** Consider $g \in \mathcal{G}$ so $g$ strictly decreases on $[a,m]$ and strictly increases on $[m,b]$. Consider a map $\tilde{f} : [a,b] \to [a,b]$, defined as $\tilde{f}(x) = a + b - x$ for $x$ in $[a,b]$. Let $f \equiv \tilde{f}^{-1} \circ g \circ \tilde{f}$. As shown in the proof of Theorem 2 in Deng and Khan (2018), $f$ and $g$ are topologically conjugate and $f \in \mathcal{F}$ strictly increases on $[a, a+b-m]$ and strictly decreases on $[a+b-m, b]$. Let $\Pi \equiv \{x \in [a+b-m, b] : f(x) \in [a+b-m, b] \text{ and } f^2(x) = x\}$. By construction, we have $\Pi = \{y : (a+b-y) \in \Pi'\}$. Since $f^2(a+b-m) = a+b-g(a+b-f(a+b-m)) = a+b-g^2(m)$, $g^2(m) > m$ is equivalent to $f^2(a+b-m) < a+b-m$. Since $f^3(a+b-m) = a+b-g(a+b-f^2(a+b-m)) = a+b-g^3(m)$, $g^3(m) \geq \min\{\Pi'\}$ is equivalent to $f^3(a+b-m) \leq \max\{\Pi\}$. Moreover, $g^3(m) > \max\{x \in \Pi'\}$ is equivalent to $f^3(a+b-m) < \min\{x \in \Pi\}$. Then the two equivalence results directly follow from Theorems 2 and 3. $\square$

**Proof of Proposition 2.** Since $m = \phi_1/(2\phi_2)$, we have

$$f_b(m) = \phi_1 \cdot \frac{\phi_1}{2\phi_2} - \phi_2 \cdot \left(\frac{\phi_1}{2\phi_2}\right)^2 + c = \frac{\phi_1^2}{4\phi_2} + c$$

$$f_b^2(m) = f_b(f_b(m)) = \phi_1 \cdot \left(\frac{\phi_1^2}{4\phi_2} + c\right) - \phi_2 \cdot \left(\frac{\phi_1^2}{4\phi_2} + c\right)^2 + c$$

$$= -\phi_2 c^2 + \left(\phi_1 - \frac{\phi_1^2}{2} + 1\right) c + \left(\frac{\phi_1^3}{4\phi_2} - \frac{\phi_1^4}{16\phi_2}\right).$$

Then, $f_b^2(m) < m$ can be written more explicitly as

$$\begin{aligned}
&-\phi_2 c^2 + \left(\phi_1 - \frac{\phi_1^2}{2} + 1\right)c + \left(\frac{\phi_1^3}{4\phi_2} - \frac{\phi_1^4}{16\phi_2}\right) < \frac{\phi_1}{2\phi_2} \\
\Leftrightarrow\ &\phi_2 c^2 - \left(\phi_1 - \frac{\phi_1^2}{2} + 1\right)c + \left(\frac{\phi_1}{2\phi_2} - \frac{\phi_1^3}{4\phi_2} + \frac{\phi_1^4}{16\phi_2}\right) > 0 \\
\Leftrightarrow\ &\frac{1}{16\phi_2}\left[4\phi_2 c + (\phi_1 - 1)^2 - 1\right]\left[4\phi_2 c + (\phi_1 - 1)^2 - 5\right] > 0
\end{aligned}$$

Since we know $4\phi_2 c + (\phi_1 - 1)^2 \geq 4$, $f_b^2(m) < m$ is equivalent to $4\phi_2 c + (\phi_1 - 1)^2 > 5$.

We now claim that $\Pi$ is a singleton if $f_b^2(m) < m$. To this end, for any $x \in X_b$,

$$\begin{aligned}
f_b^2(x) - x &= \phi_1 f_b(x) - \phi_2 (f_b(x))^2 + c - x \\
&= \phi_1 f_b(x) - \phi_2 (f_b(x))^2 + c - [f_b(x) - (f_b(x) - x)] \\
&= \phi_1 f_b(x) - \phi_2 (f_b(x))^2 + c - [\phi_1 x - \phi_2 x^2 + c - (f_b(x) - x)] \\
&= (\phi_1 + 1)(f_b(x) - x) + \phi_2(-(f_b(x))^2 + x^2) \\
&= (f_b(x) - x)[\phi_1 + 1 - \phi_2(f_b(x) + x)].
\end{aligned}$$

Thus, for any $\pi \in \Pi$, $f_b^2(\pi) - \pi = 0$ implies (i) $f_b(\pi) = \pi$ or (ii) $h(\pi) \equiv \phi_1 + 1 - \phi_2(f_b(\pi) + \pi) = 0$. From (i), we have $\pi = z$. Now consider (ii). We have

$$\begin{aligned}
h(m) =&\phi_1 + 1 - \phi_2\left(\phi_1 \cdot \frac{\phi_1}{2\phi_2} - \phi_2 \cdot \left(\frac{\phi_1}{2\phi_2}\right)^2 + c + \frac{\phi_1}{2\phi_2}\right) \\
=&\frac{1}{4}\left[5 - (4\phi_2 c + (\phi_1 - 1)^2)\right].
\end{aligned}$$

Since $f_b^2(m) < m$ implies that $4\phi_2 c + (\phi_1 - 1)^2 > 5$, $h(m) < 0$ if $f_b^2(m) < m$. Further, $h'(m) = -\phi_2(f_b'(m) + 1) = -\phi_2 < 0$ and $h''(x) = 2\phi_2^2 > 0$ for any $x \geq m$, the function $h$ first decreases from $m$ to some $x > m$ and then increases. Thus, there exists at most one root of $h(x) = 0$ for $x \geq m$. We know from the definition of $\Pi$, if there exists $\pi \in \Pi$ such that $\pi \geq m$, $\pi \neq z$, and $f_b^2(\pi) = \pi$, then $h(\pi) = 0$ and there exists another $\pi' \in \Pi$ such that $\pi' = f_b(\pi)$, $\pi' \geq m$, $\pi' \neq \pi$, $\pi' \neq z$, and $f_b^2(\pi') = \pi'$, which again imply $h(\pi') = 0$. This contradicts to the fact that there is at most one root of $h(x) = 0$ for $x \geq m$. Thus, $\Pi$ has to be a singleton if $f_b^2(m) < m$.

Since $\Pi$ is a singleton when $f_b^2(m) < m$ is satisfied, to apply Theorems 2 and 3, we just need to specialize Conditions OPC$^z$ and TUR$^z$ to the map $f_b$. Consider first $f_b^3(m) < z$ in OPC$^z$. Since $f_b(f_b^2(m)) = f_b^3(m) < z = f_b(z)$, $\phi_1 f_b^2(m) - \phi_2(f_b^2(m))^2 + c < \phi_1 z - \phi_2 z^2 + c$, which can be simplified as $(f_b^2(m) - z)(\phi_1 - \phi_2(f_b^2(m) + z)) < 0$. Since $f_b(m) > f_b(z) = z$, $f_b^2(m) = f_b(f_b(m)) < f_b(z) = z$ and then $f_b^3(m) < z$ is equivalent to $\phi_1/\phi_2 > (f_b^2(m) + z)$. Using $z = (\phi_1 - 1 + \sqrt{(\phi_1 - 1)^2 + 4\phi_2 c})/(2\phi_2)$ and the expression of $f_b^2(m)$, we can write this condition explicitly as

$$\begin{aligned}
&-\phi_2 c^2 + \left(\phi_1 - \frac{\phi_1^2}{2} + 1\right)c + \left(\frac{\phi_1^3}{4\phi_2} - \frac{\phi_1^4}{16\phi_2}\right) + \frac{\phi_1 - 1 + \sqrt{(\phi_1 - 1)^2 + 4\phi_2 c}}{2\phi_2} < \frac{\phi_1}{\phi_2} \\
\Leftrightarrow\ &\phi_2^2 c^2 - \phi_2\left(\phi_1 - \frac{\phi_1^2}{2} + 1\right)c + \left(\frac{1}{2} + \frac{\phi_1}{2} - \frac{\phi_1^3}{4} + \frac{\phi_1^4}{16}\right) - \frac{\sqrt{(\phi_1 - 1)^2 + 4\phi_2 c}}{2} > 0
\end{aligned}$$

$$\Leftrightarrow \left[(\phi_1-1)^2+4\phi_2 c\right]^2-6[(\phi_1-1)^2+4\phi_2 c]-8\sqrt{(\phi_1-1)^2+4\phi_2 c}+13>0$$

Let $t=\sqrt{(\phi_1-1)^2+4\phi_2 c}$. We further simplify the last inequality as $t^4-6t^2-8t+13=(t-1)(t^3+t^2-5t-13)>0$ and using $t\in[2,3]$, we obtain $t^3+t^2-5t-13>0$. Since $\ell(t)\equiv t^3+t^2-5t-13$ strictly increases for $t\in[2,3]$ and $\ell(2)<0$ and $\ell(3)>0$, there exists a unique root $\hat{t}\approx 2.6786$ such that $\ell(\hat{t})=0$. Thus, $f_b^3(m)<z$ is equivalent to $(\phi_1-1)^2+4\phi_2 c>\hat{t}^2$. Since $\hat{t}^2>5$, $(\phi_1-1)^2+4\phi_2 c>\hat{t}^2$ also ensures $f_b^2(m)<m$.

Thus, applying Theorem 2, we obtain the desired condition for the existence of odd-period cycles.. Similarly, we can apply Theorem 3 to obtain the necessary and sufficient condition for $f_b^2$ to be turbulent. □

**Proof of Proposition 3.** We know $m=x_c$, so $f_{ii}^2(m)=(1+g)^{1-n}x_c$ and thus, $f_{ii}^2(m)<m$ if and only if $n>1$. Solving for the interior fixed point, we obtain $z=(1+g)^{1/(n+1)}x_c$. For any $x>m$, $f_{ii}^2(x)=(1+g)^{1-n}x_c^{1-n^2}x^{n^2}$. For any $x>m$, $f_{ii}(x)=x$ implies that $x=(1+g)^{1/(n+1)}x_c=z$ where the first equality follows from $n>1$. Since we know $n>1$, we must have $f_{ii}^2(m)<m$ and $\Pi$ being a singleton. Notice that if $n=1$, $\Pi=[x_c,x_c(1+g)]$. Further, if $f_{ii}^2(m)<m$, then $f_{ii}^3(m)<z$ can be explicitly written as $f_{ii}(f_{ii}^2(m))=(1+g)^{2-n}x_c<z=(1+g)^{1/(n+1)}x_c$, which holds if and only if $2-n<1/(n+1)$, or equivalently, $n>(\sqrt{5}+1)/2$ (since $n>1$). Similarly, $f_{ii}^3(m)\le z$ if and only if $n\ge(\sqrt{5}+1)/2$. Applying Theorems 2 and 3, we obtain the desired conditions. □

**Proof of Proposition 5.** Since $m=f/(\xi+1-d)$, we have

$$f_{dj}^2(m)=\frac{[d\xi+(1-d)]f}{\xi+(1-d)}>\frac{f}{\xi+(1-d)}=m,$$

where the inequality follows from $\xi>1$. Since $z=f/(\xi+1)$, $f^3(m)>z$ can be explicitly written as

$$(1-d)\frac{[d\xi+(1-d)]f}{\xi+(1-d)}>\frac{f}{\xi+1}\Leftrightarrow d\xi+(1-d)>\frac{\xi+(1-d)}{(\xi+1)(1-d)}$$
$$\Leftrightarrow 1+(\xi-1)d>1+\frac{\xi d}{(\xi+1)(1-d)}\Leftrightarrow (\xi-1)d>\frac{\xi d}{(\xi+1)(1-d)}$$
$$\Leftrightarrow \left(\xi-\frac{1}{\xi}\right)(1-d)>1.$$

Since we know $\Pi'=\{z\}$ and $f_{dj}^2(m)>m$ always holds (since $\xi>1$), $f_{dj}$ has an odd-period cycle if and only if $f_{dj}^3(m)>z$ or equivalently, $(\xi-1/\xi)(1-d)>1$. Similarly, we can establish the sufficient and necessary condition for the turbulence of $f_{dj}^2$ to be $(\xi-1/\xi)(1-d)\ge 1$. □

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