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RESEARCH ARTICLE

On optimal extinction in the matchbox two-sector model

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Abstract

We provide a complete characterization of optimal extinction in a two-sector model of economic growth through three results, surprising in both their simplicity and intricacy. (i) When the discount factor is below a threshold identified by the well-known δ-normality condition for the existence of a stationary optimal stock, the economy's capital becomes extinct in the long run. (ii) This extinction may be staggered if and only if the investment-good sector is capital-intensive. (iii) We uncover a sequence of thresholds of the discount factor, identified by a family of rational functions, that represent bifurcations for optimal postponements on the path to extinction. We also report various special cases of the model having to do with unsustainable technologies and equal capital-intensities that showcase long-term optimal growth, all of topical interest and all neglected in the antecedent literature.

Keywords Extinction · Capital intensity · Two-sector · δ-Normality · Bifurcation

JEL Classification C60 · D90 · O21

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A formal presentation demands a precision in thinking and encourages a search for the most direct route from a set of assumptions to a conclusion. Despite its stark simplicity, a model may dramatically confirm or reject an "intuitive" perception, and may display highly complex, essentially unpredictable evolution, allowing for possibilities of extinction and indefinite sustainability. Even small changes may set the stage for inevitable *rather than* possible *extinction and emergence of "thresholds" or "tipping points" that mark a change from growth to a stunning inevitability of extinction.*[1] Majumdar (2020)

1 Introduction

The notion of a stationary capital stock, also referred to as a stationary optimal program, is central to the theory of aggregative and multi-sectoral descriptive and optimal growth, as it stems from the pioneering papers of Ramsey (1928) and von Neumann (1945).[2] Adopting a primal approach, Khan and Mitra (1986) obtain a sufficient condition concerning the discount factor and the technology, namely the δ-normality

[1] This epigraph is cobbled from several sentences: for the first two, see p. vii from the preface, and the third from pp. 25–26, all from Majumdar (2020). Section 5.4 is directly relevant to this paper. More generally, this book addresses topical issues of the day, and merits a careful study.

[2] It is now well understood that Ramsey's 1928 effort was rediscovered by Cass (1965) and Koopmans (1965), but the RCK label, common for the workhorse model of modern macroeconomics, does not acknowledge either Samuelson (1965) and its earlier multi-sectoral extension by Samuelson and Solow (1956), or the independent analysis of Malinvaud (1965); see Shell (1967) for elaborations in continuous time and the use of Pontryagin's principle; also see Spear and Young (2014, 2015) for details. Samuelson and Solow (1956) concern themselves with multi-sectoral optimal growth theory, but closely follow Ramsey, while von Neumann's contribution sits astride descriptive and optimum growth theory in that it involves maximization but not that of a Ramseyian planner. The notion of a blanced growth rate is, to be sure, directly connected to that of a stationary capital stock; see Koopmans (1964) and Burmeister (1974).

condition, for the existence of a unique non-trivial stationary optimal stock for a large class of multi-sectoral optimal growth models.[3] The literature has since largely presumed the existence of a non-trivial stationary optimal stock by explicitly or implicitly imposing this δ-normality condition. Little of the existing work investigates the non-fulfilment of the condition and its resulting implications. This paper takes up this open question not merely to close a theoretical lacuna, important though that is, but also to study the possibilities regarding issues of survival and optimal extinction of the capital stock that are opened up by the non-fulfilment of this condition.[4]

The question is best investigated in a model in which the existence of an optimal program is assured, but so is the non-existence of a non-trivial stationary capital stock; a model tractable enough for the question at hand, yet with findings whose robustness is not called into question in a fuller multi-sectoral setting. The canonical two-sector Robinson-Srinivasan-Leontief (RSL) model of optimal growth fits this need well,[5] and the results it furnishes are surprising both for their simplicity *and* their complication. This model consists of a consumption-good and an investment-good sector, and with Leontief production technologies in both sectors. The (Ramseyian) social planner maximizes the discounted sum of future utilities by allocating capital and labor between the two sectors.

Under the aforementioned δ-normality condition, more specifically, when the discount factor is above the inverse of the marginal rate of transformation (MRT) under full specialization in the investment-good sector, ($\delta > 1/\theta$), this model has been employed as a workhorse to demonstrate how a wide array of dynamics,[6] ranging from monotone convergence to cycles and chaos, arises from a simple economic model. The question then is what happens when the δ-normality condition is not fulfilled? And as befits any analysis of a two-sector model, we ask this question under different capital intensity conditions, a "casual property of the technology" being given prominence in Solow's rather immediate response to Uzawa's contribution:

> My second objective is to try to elucidate the role of the crucial capital-intensity condition in Uzawa's model. He finds that his model economy is always stable if the consumption-goods sector is more capital-intensive than the investment-goods sector. It seems paradoxical to me that such an important characteristic of the equilibrium path should depend on such a casual property of the technology. And since this stability property is the one respect in which Uzawa's results seem

[3] The state of the art result is in Section 7.5 of McKenzie (2002) where the author refers to McKenzie (1986) and to the work of Peleg-Ryder. In his Handbook survey, he cites the work of Flynn, Khan-Mitra and Sutherland; the relevant result is Theorem 7.1 which uses Lemma 7.1 ascribed to the 1984 working paper version of Khan and Mitra (1986); see the overview in the Handbook chapter of Mitra and Nishimura (2006).

[4] For the topicality, not to say immediacy of these issues, see, in addition to Majumdar (2020), Managi (2015), and their references. The latter is ostensibly phrased in the Asian context, yet testifies to the fact that the very nature of the problem spills beyond national boundaries; see for example the chapter on environment and growth by Horii and Ikefuji (2015).

[5] The RSL model can be viewed as a special case of Morishima's matchbox two-sector model. Morishima (1969) first introduces and analyzes a "Walras-type model of matchbox size", featuring Leontief production technologies in a two-sector setting. Lectures in Morishima (1965) are the natural precursor to the book.

[6] See the related literature on the two-sector growth theory documented below.

> qualitatively different from those of my 1956 paper on a one-sector model, I am anxious to track down the source of the difference.[7]

We are anxious to see what happens to issues of survival and optimal extinction when there is no non-trivial stationary optimal stock and *a fortiori,* any convergence to it is precluded at the very outset.

In broad outline, the dispensation of the δ-normality condition in the RSL model furnishes three results. First, capital stock always converges to zero in the long run when the discount factor is below $1/\theta$. For a less productive investment-good sector, this threshold ($1/\theta$) can be close to unity and thus extinction may take place even for relatively patient agents. Second, the deferment of extinction with positive investment arises *only* if the investment-good sector is more capital intensive; if less intensive, the economy fully specializes in the consumption-good sector on the path to extinction. This asymmetry stems from the fact that production of consumption goods requires relatively less capital when the investment-good sector is more capital intensive, and it is then optimal to trade off today's utility by diverting resources to investment for tomorrow's consumption gains. Third, perhaps most intriguingly, for the case of a capital-intensive investment-good sector, we identify an *infinite* sequence of thresholds for the discount factor at which the optimal policy bifurcates. As the discount factor rises, the economy will stay longer in the phase of diversification, with production resources fully utilized in consumption- and investment-good production, thus leading to a longer delay in extinction.

We further extend the characterization of optimal policy to three special cases. First, in the case of an unsustainable RSL technology ($\theta < 1$), that is, any positive capital stock being technologically unsustainable in the long run, the planner may still allocate resources to the investment-good sector along the optimal path to extinction when the investment-good sector is capital intensive. Second, when there is no difference in capital intensities, and the model reduces to the one-sector case, the optimal dynamics mirror the case of a capital-intensive consumption-good sector, bringing about extinction without investment.[8] Third, in the knife-edge case for the discount factor in which the optimal policy is no longer unique, and the optimal policy manifests itself as a correspondence, the door is opened to a variety of long-term outcomes.

Our results contribute directly to the two-sector optimal growth theory. Benhabib (1992) and Majumdar et al. (2000) still remain current as the go-to anthologies, and can be complemented by chapters in Dana et al. (2006): they emphasize the existence of cycles and chaos even when intertemporal arbitrage opportunities are precluded by the assumption of an infinitely-lived Ramseyian planner.[9] More specifically, the two-sector RSL model that this paper examines has been used as a workhorse in this

[7] See Solow (1961). Solow specifies the notion of stability that is subscribing to: "[The model economy is stable] in the sense that full employment requires an approach to a state of balanced expansion."

[8] If the felicity function is instead assumed to be strictly concave, delayed extinction with investment can also arise in the one-sector setting. Our main characterization results demonstrate investment along the extinction path even without such strict concavity in a two-sector setting. We thank our referee for this observation and will elaborate on it in Section 5.3.

[9] Cycles and complicated dynamics are also known to arise from the overlapping generations (OLG) model (Benhabib and Day 1982; Grandmont 1985). For the dynamics of two-sector OLG models, see Galor (1992) and Reichlin (1992); also see Hirano and Stiglitz (2022).

literature. In a seminal paper, Nishimura and Yano (1995) demonstrate in the RSL model with circulating capital that optimal ergodic chaos can arise even for arbitrarily patient agents. In another special case of the RSL model, the so-called Robinson-Solow-Srinivasan (RSS) model, Khan and Mitra (2005) demonstrate the emergence of optimal topological chaos.[10] The upshot of the existing work is that under the δ-normality condition, the optimal policy for the RSL model is rich and complex for the case of a capital-intensive consumption-good sector and simple and uniform for the case of a capital-intensive investment-good sector (Fujio et al. 2021). This paper demonstrates that it is this dichotomy that is reversed under the non-fulfilment of the δ-normality condition.

From a substantive point of view, this paper complements the discussions geared more towards resource economics as pioneered by Clark (1976). Cropper et al. (1979) and Cropper (1988) establish conditions for optimal extinction of renewable resources. Since the issues of survival and extinction relate to the future and future-uncertainty, they demand one to move on from the deterministic framework of optimal growth to a stochastic setting. Mitra and Roy (2006) study optimal management of natural resources under uncertainty and characterize conditions that yield extinction. Also in a stochastic one-sector growth setting, Kamihigashi (2006) offers sufficient conditions for almost sure convergence to zero stock and connect the conditions with an intuitive notion of volatility.[11]

The rest of the paper is structured as follows. We introduce the model and preliminaries in the next section. In Sect. 3, we present the results on optimal extinction without investment. In Sect. 4, we explain the construction of thresholds for the discount factor, which are then used as bifurcation values in our characterization of optimal extinction with investment. Several special cases, including a numerical example, are discussed in Sect. 5. In keeping with the epigraph, the proofs of the results require scrupulously detailed derivation, but we make do with geometry alone.[12] We conclude in Sect. 6.

2 The model and preliminaries

2.1 The model

We consider the two-sector RSL model of optimal growth with discounting. There are two sectors: a consumption-good sector and an investment-good sector. The production technology is Leontief. It requires one unit of labor and $a_C > 0$ units of capital to produce one unit of consumption good, and one unit of labor and $a_I \geq 0$ units of

[10] Fujio (2005, 2008) characterizes the dynamic properties for the RSL model but without discounting, where extinction will never take place on the optimal path. Deng et al. (2021) depart from the optimal growth paradigm and obtain eventual periodicity in the RSL setting of equilibrium growth.

[11] But this is just the tip of the iceberg: even on limiting oneself to an aggregative stochastic environment, one has a rich literature to contend with. In a pioneering paper, Stachurski (2002) provides sufficient conditions for existence and stability of a positive steady state for a stochastic model of optimal growth with unbounded shock. Nishimura and Stachurski (2005) apply an Euler equation technique to extend the stability result. Also see Kamihigashi (2007), Kamihigashi and Roy (2006, 2007), Kamihigashi and Stachurski (2014), Mitra and Sorger (2014), and Mitra and Roy (2012, 2021, 2022).

[12] All the proofs, lemmas, and additional characterization results are collected in the “Appendix”.

capital to produce $b > 0$ units of investment good. If $a_C > a_I$, the consumption good sector is more capital intensive than the investment good sector, and if $a_C < a_I$, the investment good sector is more capital intensive than the consumption good sector. If $a_C = a_I$, the model boils down to its one-sector setting. Note that we assume $a_C > 0$ because otherwise the planner would have no incentives to produce investment goods. However, we do not exclude the possibility of $a_I = 0$ which corresponds to the two-sector RSS setting as in Khan and Mitra (2005).

Labor supply is fixed and normalized to be one in each time period t. Denote the capital stock in the current period by x, the capital stock in the next period by x', and the depreciation rate of capital by $d \in (0, 1]$. The *transition possibility set* is given by

$$\Omega = \{(x, x') \in \mathbb{R}_+ \times \mathbb{R}_+ : x' - (1-d)x \geq 0, x' - (1-d)x \leq b \min\{1, x/a_I\}\},$$

where $\mathbb{R}_+$ is the set of non-negative real numbers. Denote by y the output of consumption good. For any $(x, x') \in \Omega$, we define a correspondence

$$\Lambda(x, x') = \Bigg\{ y \in \mathbb{R}_+ : y \leq \frac{1}{a_C}\left(x - \frac{a_I}{b}(x' - (1-d)x)\right) \\ \text{and } y \leq 1 - \frac{1}{b}(x' - (1-d)x) \Bigg\}.$$

A felicity function, $w : \mathbb{R}_+ \longrightarrow \mathbb{R}$, is linear and given by $w(y) = y$. The reduced form utility function, $u : \Omega \longrightarrow \mathbb{R}_+$, is defined as

$$u(x, x') = \max\{w(y) : y \in \Lambda(x, x')\}.$$

The future utility is discounted with a discount factor $\delta \in (0, 1)$. Define

$$\zeta \equiv \frac{b}{a_C - a_I} - (1-d) \tag{1}$$

to be the MRT of capital between today and tomorrow under full utilization of both production factors. Define

$$\theta \equiv \frac{b}{a_I} + (1-d) \tag{2}$$

to be the MRT when the economy fully specializes in investment-good production with zero consumption good being produced for $x \leq a_I$.[13] We then write explicitly the reduced-form utility function

$$u(x, x') = \begin{cases} \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x', & \text{for } (a_C - a_I)x' \leq ((1-d)(a_C - a_I) - b)x + a_C b \\ \frac{1-d}{b}x - \frac{1}{b}x' + 1, & \text{for } (a_C - a_I)x' \geq ((1-d)(a_C - a_I) - b)x + a_C b \end{cases} \tag{3}$$

[13] Later we will simply refer to θ as the MRT with zero consumption.

In the reduced-form utility function above, the first line stands for the case of full utilization of capital while the second line stands for the case of full employment of labor.

An *economy E* consists of a triplet (Ω, u, δ). A *program starting from* $\bar{x} \in \mathbb{R}_+$ is a sequence $\{x_t, y_t\}_{t=0}^{\infty}$ such that $x_0 = \bar{x}$ and for any non-negative integer t, $(x_t, x_{t+1}) \in \Omega$ and $y_t = \max \Lambda(x_t, x_{t+1})$. A program $\{x_t^*, y_t^*\}_{t=0}^{\infty}$ starting from $\bar{x} \in \mathbb{R}_+$ is said to be *optimal* if

$$\sum_{t=0}^{\infty} \delta^t [u(x_t, x_{t+1}) - u(x_t^*, x_{t+1}^*)] \leq 0$$

for every program $\{x_t, y_t\}_{t=0}^{\infty}$ starting from $\bar{x}$. An optimal program starting from $\bar{x} \in \mathbb{R}_+$ is said to be *a stationary optimal program* if $x_t = \bar{x}$ for any non-negative integer t. *A stationary optimal stock* $\bar{x}$ is a non-negative real number such that there is a stationary optimal program from $\bar{x}$. A stationary optimal stock $\bar{x}$ is said to be *non-trivial* if $u(\bar{x}, \bar{x}) > u(0, 0)$.

2.2 Basic geometry

Before we turn to the formal discussion of the optimal policy, we describe in this subsection the basic geometry of the RSL model. Figure 1 illustrates the transition possibility set for the case of a capital-intensive investment-good sector ($a_C < a_I$). The OD line corresponds to full specialization of the economy in the consumption-good sector. The OVL line corresponds to full specialization of the economy in the investment-good sector. The slope of the OV line is θ. The MV line corresponds to the case of full utilization of labor and capital. The slope of this line is $(-\zeta)$. When the investment-good sector is capital intensive as it is in Fig. 1, if a production plan is above the MV line, capital is fully utilized whereas there is surplus labor. If a production plan is below the MV line, labor is fully employed whereas there is excess capacity. Moreover, IC_1, IC_2, and IC_3 in orange are the indifference curves for per-period utility. Lower indifference curves are associated with higher utility.

2.3 Preliminaries

We take the dynamic programming approach in our analysis. Define the value function $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ as

$$V(x) = \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1})$$

where $\{x_t, y_t\}$ is an optimal program starting from $x_0 = x$. For each $x \in \mathbb{R}_+$, the Bellman equation

$$V(x) = \max_{x' \in \Gamma(x)} \{u(x, x') + \delta V(x')\}$$

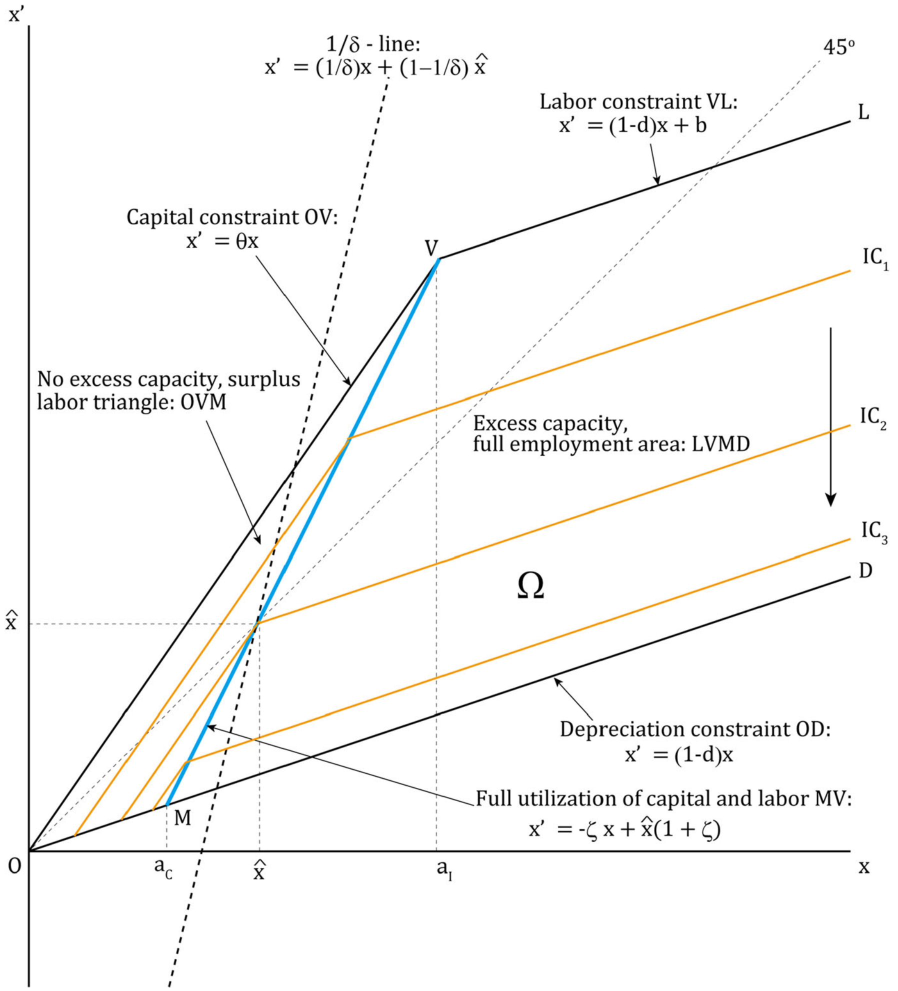

Fig. 1 The basic geometry for $a_C < a_I$ and $\delta < 1/\theta$

holds where $\Gamma(x) = \{x' : (x, x') \in \Omega\}$. For each $x \in \mathbb{R}_+$, define the *optimal policy correspondence* $h(x) = \arg\max_{x' \in \Gamma(x)}\{u(x, x') + \delta V(x')\}$. If $h(x)$ is a singleton for any $x \in \mathbb{R}_+$, then we define the *optimal policy function* $g : \mathbb{R}_+ \to \mathbb{R}_+$ as $g(x) \in h(x)$ for any $x \in \mathbb{R}_+$. A program $\{x_t, y_t\}$ from x_0 is optimal if and only if it satisfies the equation:

$$V(x_t) = u(x_t, x_{t+1}) + \delta V(x_{t+1}) \text{ for } t \geq 0.$$

The *modified golden rule* is formally defined as a pair $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$ such that $(\hat{x}, \hat{x}) \in \Omega$ and

$$u(\hat{x}, \hat{x}) + (\delta - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\delta x' - x) \text{ for all } (x, x') \in \Omega.$$

Or equivalently, the modified golden rule stock satisfies $u(\hat{x}, \hat{x}) \geq u(x, x')$ for all $(x, x') \in \Omega$ such that $x \leq (1-\delta)\hat{x} + \delta x'$. Note that $x = (1-\delta)\hat{x} + \delta x'$ corresponds to the $1/\delta$-line in Fig. 1. An economy is said to be δ-*normal* if there exists $(x, x') \in \Omega$ such that $x \leq \delta x'$ and $u(x, x') > u(0, 0)$. The following lemma provides necessary and sufficient condition for δ-normality in the RSL model.

Lemma 1 *The economy E is δ-normal if and only if $\delta > 1/\theta$.*

We now state the main existence result from Khan and Mitra (1986).

Theorem KM. *For a class $\mathcal{E}$ of qualitatively-delineated economies, if the economy is δ-normal, then there exists a modified golden-rule stock, which is also a non-trivial stationary optimal stock.*

The RSL economy E satisfies all the assumptions in Khan and Mitra (1986) and thus is in $\mathcal{E}$. We apply Theorem KM to obtain the following characterization of the modified golden rule which has been shown in Deng et al. (2019) and Fujio et al. (2021).

Proposition 1 *If $\delta > 1/\theta$, then there exists a modified golden rule given by*

$$(\hat{x}, \hat{p}) = \left(\frac{a_C b}{b + d(a_C - a_I)}, \frac{1}{(a_C - a_I)(1 + \delta\zeta)} \right),$$

and the modified golden rule stock $\hat{x}$ is the unique non-trivial stationary optimal stock.

The goal of our analysis in what follows is to characterize the optimal policy in the absence of δ-normality. Without further explicit mention, from now on we will impose the following assumption on the discount factor,

$$\delta \leq 1/\theta. \tag{4}$$

This assumption stands in sharp contrast to the assumption of $\delta > 1/\theta$ commonly imposed in the existing literature. For $\delta \leq 1/\theta$, the RSL model is no longer δ-normal and thus Theorem KM no longer applies. We will explore whether there still exists a stationary optimal stock and if not, how the economy evolves under the optimal policy. It should be noted that, in the two-sector RSS setting Khan and Mitra (2005), the investment good sector is assumed to be infinitely productive ($a_I = 0$) and as a result, this case of $\delta \leq 1/\theta$ is ruled out in the first place.

To facilitate exposition in the subsequent sections, we give a formal definition of *extinction* in the long run. We distinguish the extinction phase *without* investment, in which the economy fully specializes in consumption-good production, from the extinction phase *with* investment, in which the economy may still allocate resources to the investment-good sector despite the gradual depletion of capital stock.

Definition 1 The economy is said to be in the extinction phase without investment if the optimal policy is given by $g(x) = (1-d)x$ for any $x > 0$. The economy is said to be in the extinction phase with investment if the optimal policy yields the capital stock to converge to zero in the long run for any initial stock but there exists $x > 0$ and $x' \in h(x)$ such that $x' > (1-d)x$.

According to this definition, the economy is in the extinction phase without investment if the transition path is entirely along the OD line as in Fig. 1, and the economy is in the extinction phase with investment if the optimal policy yields depletion of capital in the long run but the transition path is not entirely along the OD line.

3 Optimal extinction without investment

We first examine the case of a capital-intensive consumption-good sector ($a_C > a_I$). It is known from the literature that, for $\delta > 1/\theta$, the optimal policy for this case involves complicated bifurcation structures and a complete characterization has not been satisfactorily obtained even for the special case of $a_I = 0$ (Khan and Mitra 2020). However, for $\delta < 1/\theta$, the optimal policy for the case of $a_C > a_I$ is surprisingly simple and uniform.

Theorem 1 *In the case of a capital-intensive consumption-good sector* ($a_C > a_I$)*, all rates of time preference* δ *less than the inverse of the MRT with zero consumption* ($\delta < 1/\theta$) *lead to an optimal policy under which the economy is in the extinction phase without investment.*

Corollary 1 *In the case of a capital-intensive consumption-good sector* ($a_C > a_I$) *and circulating capital* ($d = 1$)*, if* $\delta < 1/\theta$*, then the optimal policy yields immediate extinction:* $g(x) = 0$ *for any* $x > 0$.

From Theorem 1, there does not exist a non-trivial stationary optimal stock for $\delta < 1/\theta$. As illustrated in Fig. 2, the optimal policy is represented by the OD line: The economy converges monotonically to extinction ($x = 0$) with no investment along the optimal path. Corollary 1 further suggests that if capital is circulating ($d = 1$), capital will be depleted just in one period.

We now turn to the case of a capital-intensive investment-good sector ($a_C < a_I$). We first define

$$\mu_0 \equiv \frac{1}{\frac{b}{a_C} + (1-d)} < \frac{1}{\frac{b}{a_I} + (1-d)} = \frac{1}{\theta},$$

where the inequality follows from $a_C < a_I$. It is worth noting that from the formula above, there is a direct parallelism between μ_0 and $1/\theta$.

Theorem 2 *In the case of a capital-intensive investment-good sector* ($a_I > a_C$)*, all rates of time preference* δ *less than a technological upper bound* ($\delta < \mu_0$) *lead to an optimal policy under which the economy is in the extinction phase without investment.*

Corollary 2 *In the case of a capital-intensive investment-good sector* ($a_I > a_C$) *and circulating capital* ($d = 1$)*, if* $\delta < \mu_0$*, then the optimal policy yields immediate extinction:* $g(x) = 0$ *for any* $x > 0$.

Theorem 2 says that if the discount factor is sufficiently low, the optimal policy for the case of $a_C < a_I$, represented by the OD line in Fig. 1, is the same as that

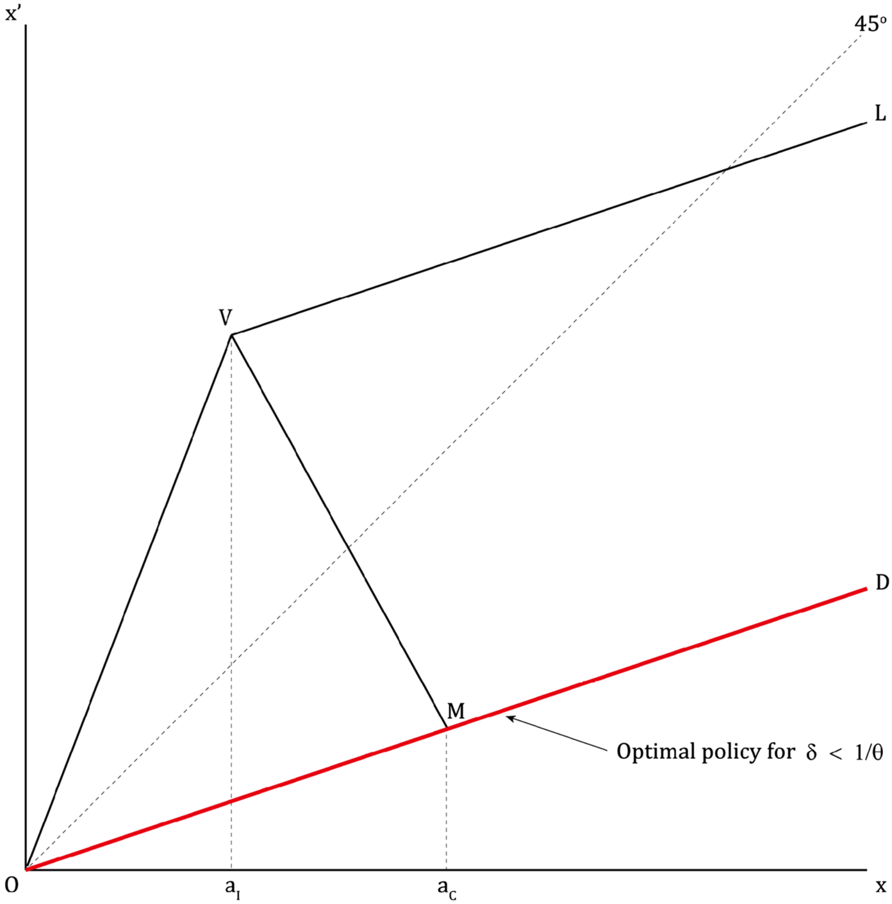

Fig. 2 The optimal policy for $a_C > a_I$ and $\delta < 1/\theta$

for $a_C > a_I$. Theorems 1 and 2 underscore that impatience leads to extinction: The economy fully specializes in the consumption-good sector if agents are sufficiently impatient. Then, what remains open is for the discount factor between μ_0 and $1/\theta$ in the case of a capital-intensive investment-good sector. This is what we turn to next.

4 Optimal extinction with investment

In the case of a capital-intensive investment-good sector ($a_I > a_C$), we will show that if the discount factor δ is in $(\mu_0, 1/\theta)$, the economy will converge to extinction in the long run but with positive investment along the transition path. The optimal policy bifurcates with respect to the discount factor in a rather intriguing manner. To characterize the optimal policy and its bifurcation structure, we first introduce a sequence of thresholds for the discount factor.

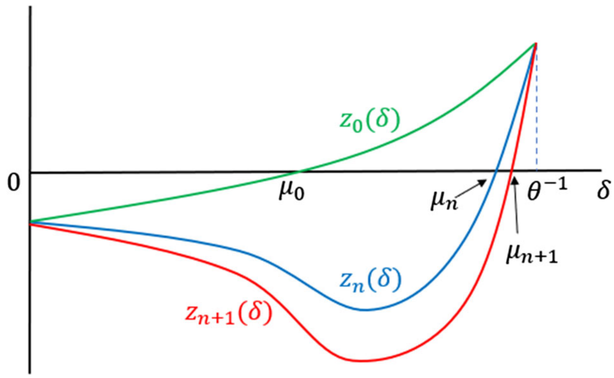

Fig. 3 The single crossing property of $z_n(\cdot)$

4.1 Thresholds for the discount factor

To define a sequence of thresholds for the discount factor $\delta \in [\mu_0, 1/\theta)$, we consider, for any natural number n, the following rational function from $[0, 1/\theta]$ to $\mathbb{R}$ as

$$z_n(\delta) \equiv -\frac{1}{b} + \delta \left(-\frac{\sum_{i=0}^{n-1} (-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^n}{a_C(1 - \delta(1-d))} \right). \tag{5}$$

Further, we define

$$z_0(\delta) \equiv -\frac{1}{b} + \frac{\delta}{a_C(1 - \delta(1-d))}, \tag{6}$$

which admits a unique root over the interval $[0, 1/\theta]$ given by μ_0 as defined in the last section. The function $z_n(\cdot)$ plays a central role in the establishment of the optimal policy. The following two lemmas state some useful properties of $z_n(\cdot)$.

Lemma 2 *Let $a_I > a_C$. For any non-negative integer n, there exists $\mu_n \in (0, 1/\theta)$ such that (i) $z_n(\mu_n) = 0$; (ii) $z_n(\delta) < 0$ for $\delta \in [0, \mu_n)$; (iii) $z_n(\delta) > 0$ for $\delta \in (\mu_n, 1/\theta]$.*

Lemma 3 *Let $a_I > a_C$. For any $\delta \in (0, 1/\theta)$ and any natural number n, $z_n(\delta) < z_{n-1}(\delta)$.*

The qualitative features of $z_n(\cdot)$ are illustrated in Fig. 3. As shown in Lemma 2, $z_n(\cdot)$ has an important "single-crossing" property on $[0, 1/\theta]$. The curve for $z_n(\cdot)$, starting from $z_n(0) < 0$ and ending at $z_n(1/\theta) > 0$, always cross the horizontal axis only once, which guarantees a unique root. Moreover, according to Lemma 3, for any non-negative integer n, the curve of $z_{n+1}(\cdot)$ always lies below that of $z_n(\cdot)$, which further suggests the monotonicity of the root associated with $z_n(\cdot)$ with respect to n. Based on the properties of $z_n(\cdot)$ stated in Lemmas 2 and 3, we can prove the following proposition.

Proposition 2 *Let $a_I > a_C$. For any $n \in \mathbb{N}$, there exists a unique root of $z_n(\delta) = 0$ for $\delta \in (0, 1/\theta)$, denoted by μ_n. The sequence $\{\mu_n\}_{n=0}^{\infty}$ satisfies (i) $\mu_n > \mu_{n-1}$ for any $n \in \mathbb{N}$ and (ii) $\lim_{n\to\infty} \mu_n = 1/\theta$.*

According to Proposition 2, there is a unique $\mu_n \in (0, 1/\theta)$ such that $z_n(\mu_n) = 0$. The family of rational functions $\{z_n(\cdot)\}_{n=0}^{\infty}$ then yield a well-defined sequence of technological parameters $\{\mu_n\}_{n=0}^{\infty}$. This sequence starts from μ_0, is strictly increasing, and converges to $1/\theta$. In what follows, we will demonstrate this sequence to be the thresholds of the discount factor at which the optimal policy bifurcates.

4.2 Optimal delays in extinction: bifurcation results

In this subsection, we state the main theorem for extinction with investment for the case of $\theta \geq 1$, under which the economy can sustain a positive level of capital stock in the long run provided that a sufficient amount of recourse is allocated to the investment-good sector. The optimal policy for the (neglected) case of $\theta < 1$, under which capital stock depletes in the long run even when the economy fully specializes in investment-good production, is qualitatively similar and will be discussed in the next section on the special cases of the model.

To ease the exposition of our characterization results, we define another sequence $\{x_n\}_{n=0}^{\infty}$ of thresholds for capital stock as follows: $x_0 \equiv a_C$ and for any $n \in \mathbb{N}$,

$$x_n = -\frac{1}{\zeta}\left(x_{n-1} - \frac{a_C b}{a_C - a_I}\right). \tag{7}$$

We illustrate the construction of this sequence in Fig. 4. The sequence starts from $x_0 = a_C$. Given our construction, for any $n \in \mathbb{N}$, (x_n, x_{n-1}) is on the MV line where capital and labor are fully utilized. Geometrically, it is clear that this sequence converges to $\hat{x}$.

Lemma 4 *Let $a_C < a_I$. The sequence $\{x_n\}_{n=0}^{\infty}$ is monotonically increasing: $x_n > x_{n-1}$ for any $n \in \mathbb{N}$. Further, $\lim_{n\to\infty} x_n = \hat{x}$ for $\theta > 1$ and $\lim_{n\to\infty} x_n = a_I$ for $\theta = 1$.*

Lemma 4 states formally the monotonicity and the limit of $\{x_n\}_{n=0}^{\infty}$ for $\theta \geq 1$.[14] With Lemma 4 and Proposition 2, we are ready to present the main characterization result for extinction with investment. The next proposition summarizes the bifurcation structure of the optimal policy with respect to the discount factor δ for $\delta \in (\mu_0, 1/\theta)$. To bring out the most salient bifurcation pattern, we focus on the case of δ strictly between two consecutive thresholds. In the "Appendix", we present the additional characterization results for $\delta = \mu_n$ for which the optimal policy becomes a correspondence.

[14] Recall $\hat{x} = a_C b/(b + d(a_C - a_I))$, so the limit of $\{x_n\}_{n=0}^{\infty}$ can also be uniformly written as $\lim_{n\to\infty} x_n = a_C b/(b + d(a_C - a_I))$ for both $\theta < 1$ and $\theta = 1$.

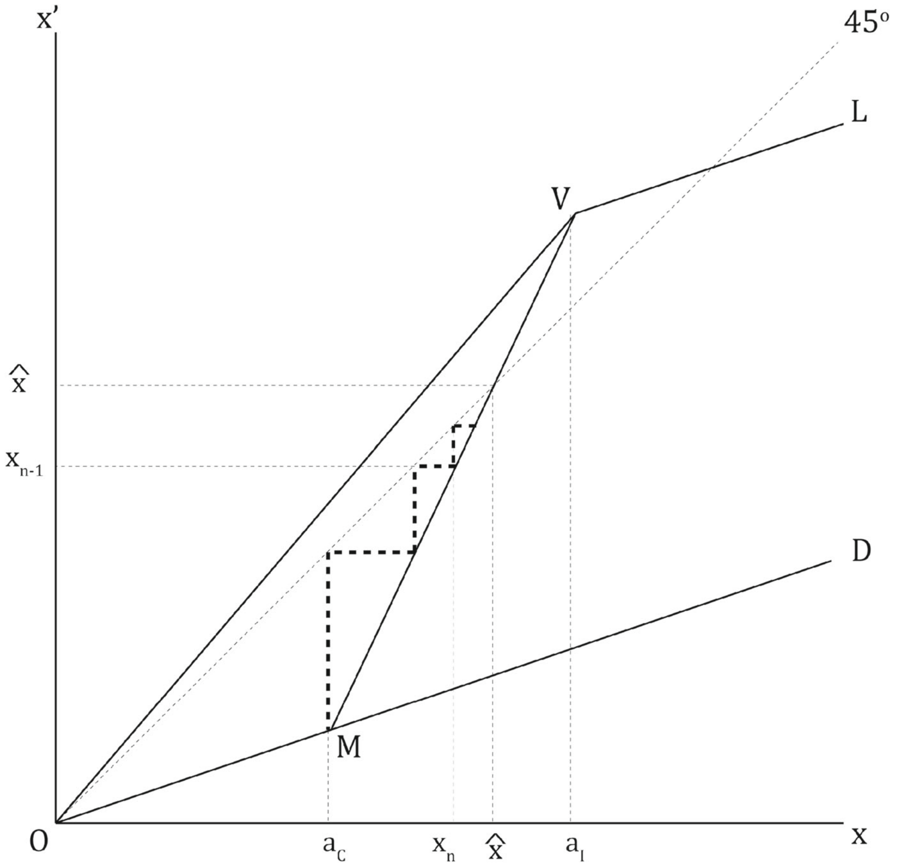

Fig. 4 The construction of x_n for $a_C < a_I$

Proposition 3 *Let $a_C < a_I$, $\theta \geq 1$, and $0 < d < 1$. If $\mu_{n-1} < \delta < \mu_n$ for $n \in \mathbb{N}$, then the optimal policy function is given by*

$$g(x) = \begin{cases} (1-d)x & \text{for } x \in (0, a_C] \\ -\zeta x + \frac{a_C b}{a_C - a_I} & \text{for } x \in (a_C, x_n] \\ x_{n-1} & \text{for } x \in (x_n, \frac{x_{n-1}}{1-d}] \\ (1-d)x & \text{for } x \in (\frac{x_{n-1}}{1-d}, \infty) \end{cases}.$$

Figure 5 shows how the optimal policy changes with the discount factor. The first panel corresponds to the case covered by Theorem 2. The second panel plots the optimal policy for $\delta \in (\mu_0, \mu_1)$. The policy deviates from the OD line for $x \in (a_C, a_C/(1-d))$. For $x \in (a_C, x_1]$, the planner chooses to fully utilize the resources, and for $x \in (x_1, a_C/(1-d))$, the planner targets the investment-good production at a level such that capital stock tomorrow equals exactly a_C. Under this policy, for any initial stock above a_C, the economy deviates from the OD line for exactly one period along its transition path.

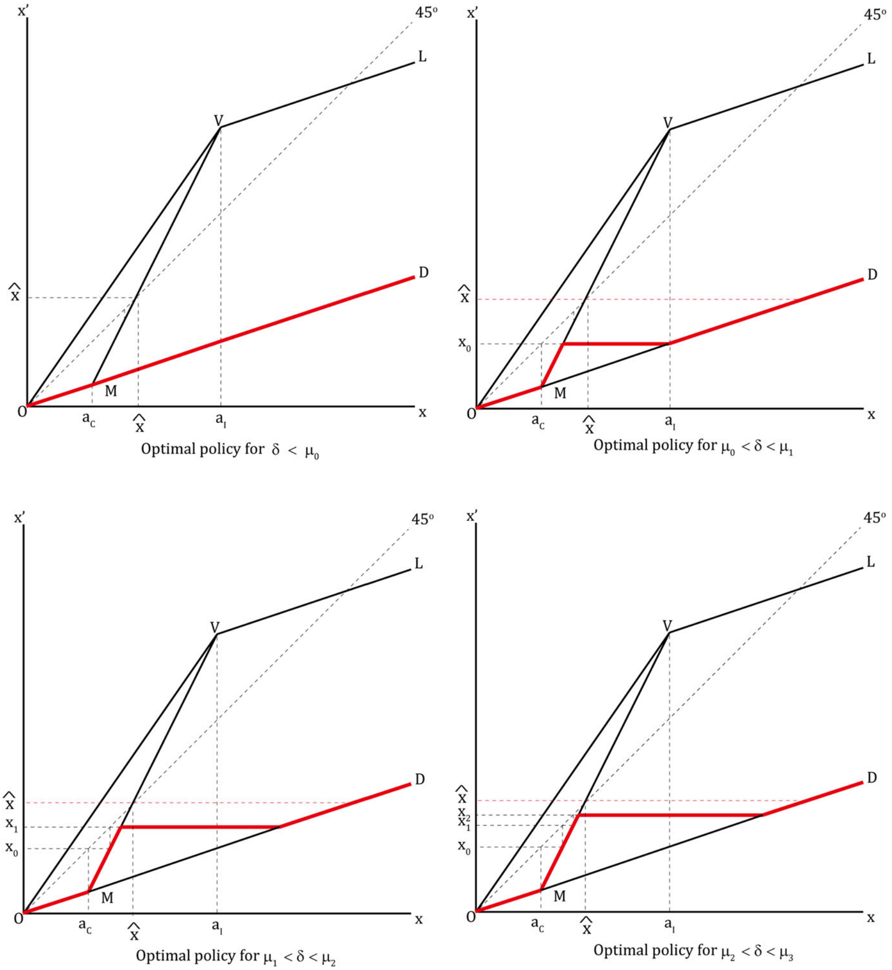

Fig. 5 The optimal policy for $a_C < a_I$, $\theta > 1$ and $0 < d < 1$

The third and fourth panel of Fig. 5 illustrate the optimal policy for the discount factor in (μ_1, μ_2) and that in (μ_2, μ_3), respectively. The interval for capital stock at which the investment-good sector is activated enlarges as the discount factor increases, but the qualitative features of the transition dynamics remain the same: For $\delta \in (\mu_0, 1/\theta)$, the optimal policy always consists of four segments, the middle two of which correspond to the case of positive investment. Moreover, for any positive integer n, if the discount factor is in (μ_{n-1}, μ_n), the economy will deviate from the OD line by producing the investment goods for exactly n periods. Since we know from Proposition 2 that the entire sequence $\{\mu_n\}_{n=0}^{\infty}$ is strictly increasing and converges to $1/\theta$, there are infinitely many bifurcations with respect to the discount factor. As the discount factor converges to $1/\theta$, the horizontal segment of the optimal policy will also approach the modified golden rule stock level $\hat{x}$, leading to more periods of delay in extinction.

Proposition 4 *Let $a_C < a_I$, $\theta \geq 1$, and $d = 1$. If $\mu_{n-1} < \delta < \mu_n$ for $n \in \mathbb{N}$, then the optimal policy function is given by*

$$g(x) = \begin{cases} 0 & \text{for } x \in (0, a_C] \\ -\zeta x + \frac{a_C b}{a_C - a_I} & \text{for } x \in (a_C, x_n] \\ x_{n-1} & \text{for } x \in (x_n, \infty) \end{cases}.$$

To bring out optimal delays in extinction in its starkest form, we present the optimal policy for circulating capital ($d = 1$) in Proposition 4. From Corollary 2, we know the optimal policy yields immediate extinction for $\delta < \mu_0$. For $\delta \in (\mu_0, 1/\theta)$, as shown in Fig. 6, the economy produces investment goods for any $x > a_C$. The higher the discount factor is, the more periods the economy will sustain full utilization of resources (on the MV line) during the transition dynamics. In particular, for any initial stock above x_n and any positive integer n, if the discount factor is in (μ_{n-1}, μ_n), the economy will produce x_{n-1} units of investment goods in the first period, then stay on the phase of full utilization of production resources for $(n-1)$ periods, and reach the state of extinction after that.

From Proposition 2, the interval $[\mu_0, 1/\theta)$ can be partitioned into $\{[\mu_{n-1}, \mu_n)\}_{n=1}^{\infty}$, so the following theorem follows immediately from the characterization results above.

Theorem 3 *In the case of a capital-intensive investment-good sector ($a_I > a_C$) and a positive capital stock being potentially sustainable ($\theta \geq 1$), all rates of time preference δ between the two technological bounds ($\mu_0 \leq \delta < 1/\theta$) lead to an optimal policy under which the economy is in the extinction phase with investment.*

Theorem 3 and the results in the previous section point to an important asymmetry: investment along the transition path to extinction can possibly occur *only* in the case of a capital-intensive investment-good sector. To understand the source of this asymmetry, we consider the following intertemporal decision. Let capital stock today x to be slightly above a_C such that in the absence of any investment, capital stock tomorrow $x' = (1-d)x$ falls under a_C. Suppose the planner deviates from full specialization in consumption goods to allocate infinitesimal amount of resources to investment. Given $x > a_C$ and investment being infinitesimal, the economy is still in the region of excess capacity and thus the marginal cost of investment in terms of the consumption goods today is given by $1/b$. We show that regardless of the capital intensity condition, for $\delta < 1/\theta$, it is optimal for the economy to specialize in the consumption-good sector when capital stock is below a_C and there is excess supply of labor. Thus, for $x' < a_C$, the economy enters the extinction phase without investment and the marginal return to investment is given by

$$\delta\left(\frac{1}{a_C} + \frac{\delta(1-d)}{a_C} + \frac{\delta^2(1-d)^2}{a_C} + \cdots\right) = \frac{\delta}{a_C(1-\delta(1-d))}.$$

When the consumption-good sector is capital intensive ($a_C > a_I$), for $\delta < 1/\theta$,

$$\frac{\delta}{a_C(1-\delta(1-d))} < \frac{1}{a_C(\theta-(1-d))} = \frac{a_I}{a_C} \cdot \frac{1}{b} < \frac{1}{b},$$

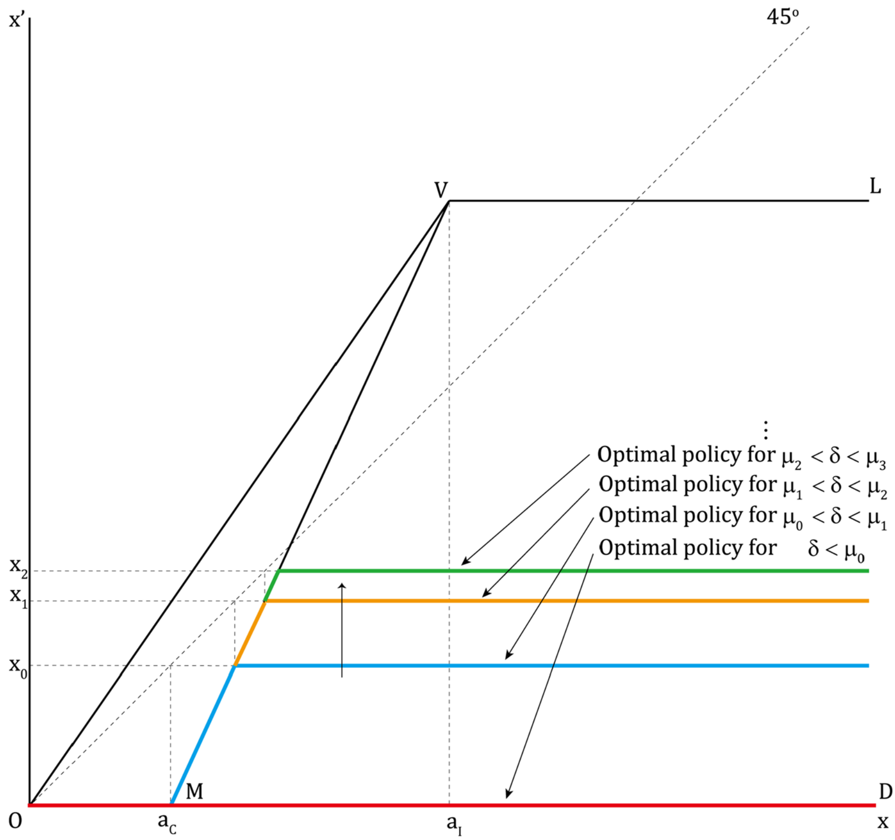

Fig. 6 The optimal policy for $a_C < a_I$, $\theta > 1$, and $d = 1$

where the second inequality follows from $a_C > a_I$, which implies the marginal cost of investment exceeds the marginal return. In contrast, when the investment-good sector is capital intensive, for $\delta \in (\mu_0, 1/\theta)$,

$$\frac{\delta}{a_C(1 - \delta(1 - d))} > \frac{1}{b}.$$

Because it requires relatively less capital to produce consumption goods for $a_C < a_I$, the marginal return to investment can potentially exceed the marginal cost. As a result, optimal extinction with investment emerges in the case of a capital-intensive investment-good sector. Further, for a larger discount factor within the interval of $(\mu_0, 1/\theta)$, the planner is more patient and thus has more incentives to invest, which translates into more periods of delay in extinction.

Moreover, the results on extinction in Theorems 1–3 yield a uniform condition of the non-existence of a non-trivial stationary optimal stock:[15] when the δ-normality

[15] The special cases of $\theta < 1$ and $a_C = a_I$ are covered in the next section.

condition is *strictly* violated ($\delta < 1/\theta$), the only stationary optimal stock that the model admits is the zero stock. The following proposition formally states this result.

Proposition 5 *For $\delta < 1/\theta$, there does not exist a non-trivial stationary optimal stock.*

5 Optimal policy: some special cases

5.1 The unsustainable technology case: $\theta < 1$

We now consider the case of $\theta < 1$. In this case, regardless of the investment decision, it is technologically infeasible to sustain any positive capital stock in the long run and extinction is guaranteed for any discount factor. Since the existing literature assumes the fulfillment of the δ-normality condition with $\delta > 1/\theta$, which requires $\theta > 1$, this unsustainable technology case has largely been neglected. Since Theorem 1 applies to both $\theta \geq 1$ and $\theta < 1$, we focus on the case of a capital-intensive investment-good sector. Our next proposition establishes the possibility of deferred extinction for this neglected case.

Proposition 6 *Let $a_C < a_I$, $\theta < 1$, and $0 < d < 1$.*

(i) *If $\mu_0 \geq 1$, the optimal policy function is given by $g(x) = (1-d)x$ for any x.*
(ii) *If $\mu_0 < 1$, there exists $n_0 \in \mathbb{N}$ such that $\mu_{n_0-1} < 1 \leq \mu_{n_0}$. For $\delta \leq \mu_{n_0-1}$, characterization of the optimal policy follows the case of $\theta \geq 1$. For $\mu_{n_0-1} < \delta < 1$, the optimal policy function is given by*

$$g(x) = \begin{cases} (1-d)x & \text{for } x \in (0, a_C] \\ -\zeta x + \frac{a_C b}{a_C - a_I} & \text{for } x \in (a_C, x_{n_0}] \\ x_{n_0-1} & \text{for } x \in (x_{n_0}, \frac{x_{n_0-1}}{1-d}] \\ (1-d)x & \text{for } x \in (\frac{x_{n_0-1}}{1-d}, \infty) \end{cases}.$$

According to Proposition 6, even the investment-good sector is highly unproductive, as long as the technological lower bound μ_0 and the discount factor satisfy $\mu_0 < \delta < 1$, the social planner would still have the incentive to allocate resources to the investment-good sector along the transition path to extinction. Qualitatively, the main difference between this case and the benchmark case of $\theta \geq 1$ in the previous section is that there are only a *finite* number of bifurcations of the optimal policy with respect to the discount factor for $\theta < 1$. Figure 7 illustrates the bifurcation structure for the case of $n_0 = 3$. Since $\mu_3 \geq 1 > \mu_2$, there are three bifurcation values for the discount factor, μ_0, μ_1 and μ_2. For $\delta > \mu_2$, the optimal policy is always represented by OMV_3M_3D. In the next proposition, we extend the result above to the case of circulating capital.

Proposition 7 *Let $a_C < a_I$, $\theta < 1$, and $d = 1$. If $\mu_0 \geq 1$, the optimal policy function is given by $g(x) = 0$ for $x > 0$. If $\mu_0 < 1$, there exists $n_0 \in \mathbb{N}$ such that $\mu_{n_0-1} < 1 \leq \mu_{n_0}$. If $\delta \leq \mu_{n_0-1}$, characterization of the optimal policy follows the*

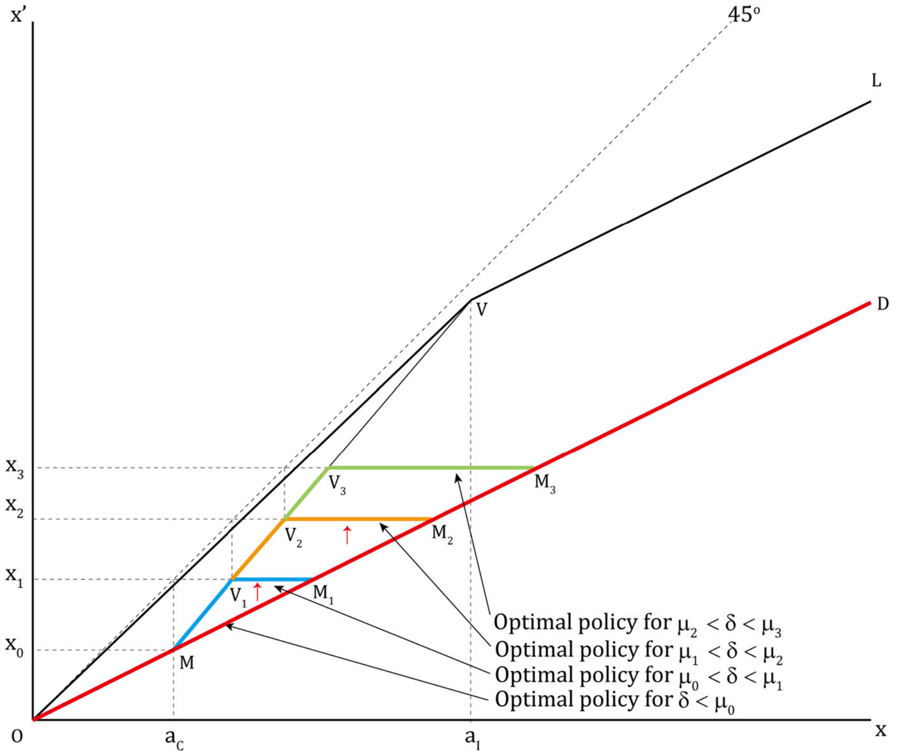

Fig. 7 The optimal policy for $a_C < a_I$ and $\theta < 1$

case of $\theta \geq 1$. *If* $\mu_{n_0-1} < \delta < 1$, *the optimal policy function is given by*

$$g(x) = \begin{cases} 0 & \textit{for } x \in (0, a_C] \\ -\zeta x + \frac{a_C b}{a_C - a_I} & \textit{for } x \in (a_C, x_{n_0}] \\ x_{n_0-1} & \textit{for } x \in (x_{n_0}, \infty) \end{cases} .$$

To summarize the bifurcation structure for the case of a capital-intensive investment-good sector, Fig. 8 illustrates the ordering of the thresholds for the discount factor with respect to $1/\theta$ and 1. There are generically two possibilities. For $\theta > 1$, the unit interval can be partitioned into three regions. The middle region contains the sequence $\{\mu_n\}_{n=0}^{\infty}$, which gives rise to infinite bifurcations. For $\theta < 1$, only a finite number of elements in the sequence will be in the unit interval, leading to finite bifurcations. It should be noted that the second panel of Fig. 8 is based on the assumption of $\mu_0 < 1$. It is also possible to have $\mu_0 \geq 1$, in which case there is extinction without investment for any discount factor.

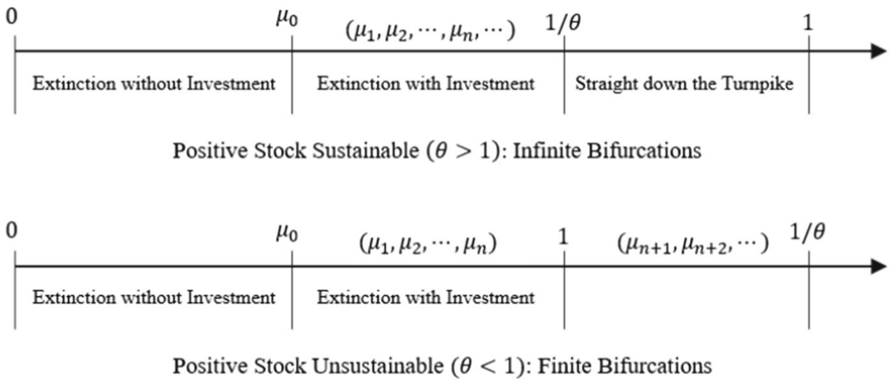

Fig. 8 The sequence $\{\mu_n\}_{n=0}^{\infty}$ and $1/\theta$ for $a_C < a_I$

5.2 The knife-edge case for the discount factor: $\delta = 1/\theta$

In this subsection, we present the results concerning an important bifurcation value for the discount factor, $\delta = 1/\theta$. For this knife-edge case, the optimal policy becomes a correspondence and there exists a continuum of non-trivial stationary optimal stocks.

Proposition 8 *Let $a_C > a_I$, $\theta > 1$, and $\delta = 1/\theta$. Then the optimal policy correspondence is given by*

$$h(x) = \begin{cases} [(1-d)x, \min\{a_C, \theta x\}] & \textit{for } x \in (0, a_I] \\ \left[(1-d)x, \min\left\{a_C, -\zeta x + \frac{a_C b}{a_C - a_I}\right\}\right] & \textit{for } x \in (a_I, a_C] \\ \{(1-d)x\} & \textit{for } x \in (a_C, \infty) \end{cases}$$

Proposition 9 *Let $a_C < a_I$, $\theta > 1$, and $\delta = 1/\theta$. The optimal policy correspondence is given by*

$$h(x) = \begin{cases} [\max\{(1-d)x, -\zeta x + \frac{a_C b}{a_C - a_I}\}, \theta x] & \textit{for } x \in (0, \frac{\hat{x}}{\theta}] \\ [\max\{(1-d)x, -\zeta x + \frac{a_C b}{a_C - a_I}\}, \hat{x}] & \textit{for } x \in (\frac{\hat{x}}{\theta}, \hat{x}] \\ \{\max\{\hat{x}, (1-d)x\}\} & \textit{for } x \in (\hat{x}, \infty) \end{cases} .$$

Propositions 8 and 9 present the optimal policy correspondence for $a_C > a_I$ and $a_C < a_I$, respectively. Figure 9 illustrates the optimal policy for both cases, in which the shaded area in red represents the optimal policy being non-unique. In particular, for both cases, any capital stock in $(0, \hat{x}]$ is a non-trivial stationary optimal stock, thus testifying that δ-normality is not a necessary condition for the existence of a non-trivial stationary optimal stock.

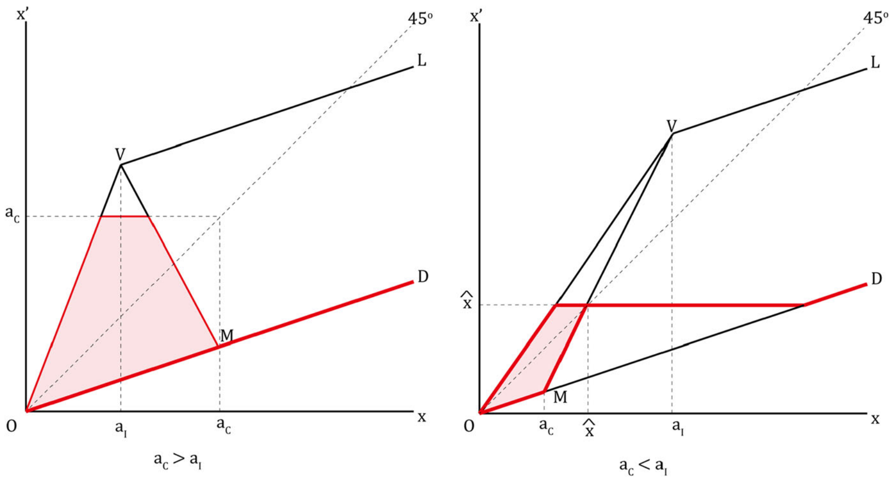

Fig. 9 The optimal policy for $\delta = 1/\theta$

5.3 The one-sector case: $a_C = a_I$

We now consider the optimal policy for the case of two sectors having the same capital intensity ($a_C = a_I$), which resembles a one-sector economy. The optimal policy for this case follows closely that for the case of a capital-intensive consumption-good sector ($a_C > a_I$), with a slight difference for $\delta = 1/\theta$, as summarized in the following proposition.

Proposition 10 *Consider the one-sector case ($a_C = a_I$). If $\delta < 1/\theta$, then the optimal policy function is given by $g(x) = (1-d)x$ for any $x > 0$. If $\delta = 1/\theta$, then the optimal policy correspondence is given by*

$$h(x) = \begin{cases} [(1-d)x, \min\{a_C, \theta x\}] & \text{for } x \in (0, a_C] \\ [(1-d)x, \max\{a_C, (1-d)x\}] & \text{for } x \in (a_C, \infty) \end{cases}$$

The result of this one-sector case can be usefully compared with Theorem 3. In the one-sector case, when the δ-normality condition is strictly violated ($\delta < 1/\theta$), there is no investment along the path to extinction. This is because the felicity function is assumed to be linear and as a result, the planner has no incentives to delay extinction for intertemporal consumption smoothing. If instead we impose strict concavity on the felicity function in this one-sector setting, delayed extinction can arise. In contrast, Theorem 3 obtains delayed extinction in the two-sector setting even without strict concavity. Our results taken together suggest that beyond consumption smoothing, there is another important *technological* reason for investment along the extinction path.

5.4 A numerical example

We finally consider a numerical example of how the optimal policy bifurcates with respect to the discount factor in the case of a capital-intensive investment-good sector. Let $b = 1$, $a_C = 2/3$, $a_I = 4/3$, and $d = 1/2$. From Eqs. (1) and (2), we have $\theta = 5/4$ and $\zeta = -2$. Then, from Eqs. (5) and (6), we have

$$
\begin{aligned}
z_0(\delta) &= -\frac{1}{b} + \delta\left(\frac{1}{a_C(1-\delta(1-d))}\right) = -1 + \frac{3\delta}{2-\delta},\\
z_1(\delta) &= -\frac{1}{b} + \delta\left(-\frac{1}{a_I - a_C} - \frac{\delta\zeta}{a_C(1-\delta(1-d))}\right) = -1 + \delta\left(-\frac{3}{2} + \frac{6\delta}{2-\delta}\right),\\
z_2(\delta) &= -\frac{1}{b} + \delta\left(-\frac{1-\delta\zeta}{a_I - a_C} + \frac{(\delta\zeta)^2}{a_C(1-\delta(1-d))}\right)\\
&= -1 + \delta\left(-\frac{3+6\delta}{2} + \frac{12\delta^2}{2-\delta}\right),
\end{aligned}
$$

which yield $\mu_0 = 1/2$, $\mu_1 = 2/3$, $\mu_2 \approx 0.73$, the first three bifurcation values for the discount factor, and we know $\lim_{n\to\infty} \mu_n = 1/\theta = 4/5$. From Eq. (7), we obtain $x_0 = a_C = 2/3$, $x_1 = 5/6$, and $x_2 = 11/12$. The optimal policy functions for $\delta = 0.4 \in (0, \mu_0)$, $\delta = 0.6 \in (\mu_0, \mu_1)$, and $\delta = 0.7 \in (\mu_1, \mu_2)$ are plotted in Fig. 10.

6 Concluding remarks

In summary, we provide a complete and comprehensive characterization of optimal policy for the two-sector RSL model in the absence of δ-normality: a categorization of extinction when the discount factor is below the MRT with zero consumption ($\delta \leq 1/\theta$). For $\delta < 1/\theta$, the optimal policy always yields extinction without investment along the transition path in the case of a capital-intensive consumption-good sector, whereas an intricate bifurcation structure emerges in the case of a capital-intensive investment-good sector. If the investment-good sector is capital intensive, and if the discount factor is between two technological bounds ($\mu_0 < \delta < 1/\theta$), the planner needs to allocate resources to the investment-good sector with resources sometimes being fully utilized so that extinction can be deferred. The results are easy to state, but difficult to obtain.

Not surprisingly, our investigation leaves several questions open. For one thing, the results of this paper lead us to pose the question as to the optimal policy for the unsustainable case ($\theta < 1$) in the RSL model without discounting. It is to us natural to pose survival and extinction issue in an undiscounted setting: if Ramsey's hesitations regarding discounting apply anywhere, they do so here. Second, from an abstract theoretical point of view, one could view the RSL model as an exalted example.[16] Because of the linear felicity function, the motive of inter-temporal

[16] We thank our referee for the encouragement to elaborate on how our results may continue to hold in more general settings.

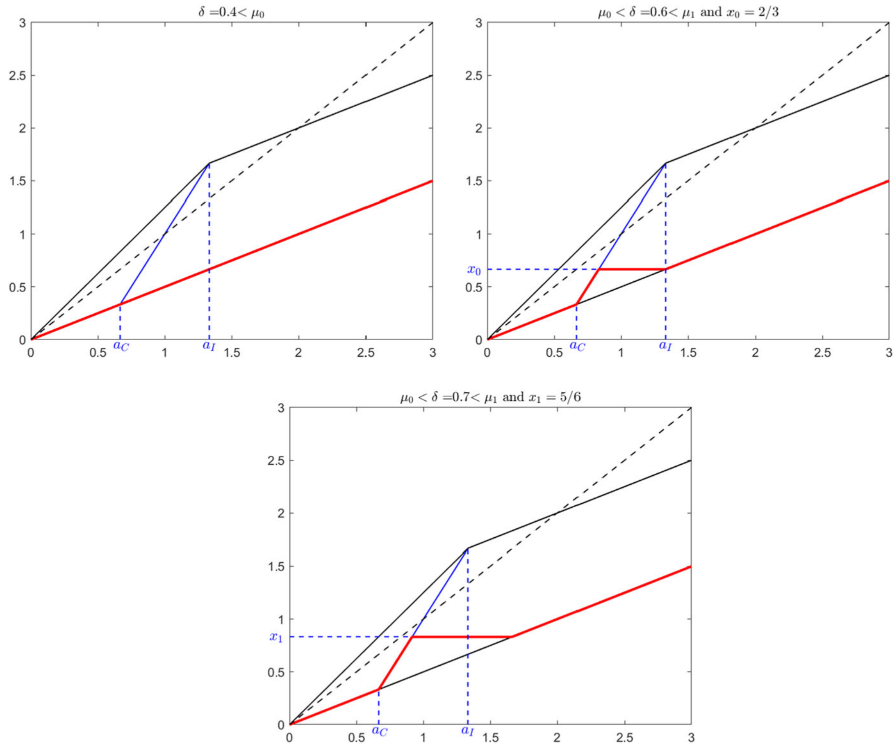

Fig. 10 A numerical example ($b = 1$, $a_C = 2/3$, $a_I = 4/3$, $d = 1/2$)

consumption smoothing is absent in our current setting, and thus the reason behind investment along the extinction path is entirely a technological one. If we relax the linearity assumption of the felicity function, we would expect that with consumption smoothing, the qualitative difference in investment along the extinction path to be less stark between the case of a capital-intensive consumption-good sector and that of a capital-intensive investment-good sector. Perhaps more interesting is to investigate how extinction with investment hinges on the Leontief specification of the production functions. We speculate that the bifurcation structure concerning the delay in extinction that we identify in the paper would continue to emerge in a setting with a more general two-sector technological specification. However, once we activate Inada conditions, the sharp specialization patterns, together with the regimes of excess capacity and unemployment, might disappear along the transition path. Finally, it is of interest to see how the results presented in this paper survive in a game-theoretic setting with many interacting agents, as in the insightful example of Mitra and Sorger (2014). We plan to take up those open questions in future research.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

A Appendix

We organize the Appendix in three parts. We first present additional characterization results on the optimal policy correspondence. We then provide the proofs of all the main results presented in the paper. Last, we present and prove lemmas that are used in the proofs of the main results.

A.1 Further characterization results

We present two additional characterization results on the optimal policy when the discount factor is equal to a cutoff value μ_n. Like the knife-edge case we identify in the paper for $\delta = 1/\theta$, the optimal policy becomes a correspondence. Proposition A1 concerns the case of durable capital ($0 < d < 1$) and Proposition A2 concerns the case of circulating capital ($d = 1$).

Proposition A1 *Let $a_C < a_I$, $\theta \geq 1$, and $0 < d < 1$.*

(i) *If $\delta = \mu_0$, then the optimal policy correspondence is given by*

$$h(x) = \begin{cases} \{(1-d)x\} & \text{for } x \in (0, a_C] \\ [(1-d)x, -\zeta x + \frac{a_C b}{a_C - a_I}] & \text{for } x \in (a_C, x_1] \\ [(1-d)x, a_C] & \text{for } x \in (x_1, \frac{a_C}{1-d}] \\ \{(1-d)x\} & \text{for } x \in (\frac{a_C}{1-d}, \infty) \end{cases}.$$

(ii) *If $\delta = \mu_n$ for $n \in \mathbb{N}$, then the optimal policy correspondence is given by*

$$h(x) = \begin{cases} \{(1-d)x\} & \text{for } x \in (0, a_C] \\ \{-\zeta x + \frac{a_C b}{a_C - a_I}\} & \text{for } x \in (a_C, x_n] \\ [x_{n-1}, \min\{-\zeta x + \frac{a_C b}{a_C - a_I}, x_n\}] & \text{for } x \in (x_n, \frac{x_{n-1}}{1-d}] \\ [(1-d)x, \min\{-\zeta x + \frac{a_C b}{a_C - a_I}, x_n\}] & \text{for } x \in (\frac{x_{n-1}}{1-d}, \frac{x_n}{1-d}] \\ \{(1-d)x\} & \text{for } x \in (\frac{x_n}{1-d}, \infty) \end{cases}.$$

Proposition A2 *Let $a_C < a_I$, $\theta \geq 1$, and $d = 1$.*

(i) *If $\delta = \mu_0$, then the optimal policy correspondence is given by*

$$h(x) = \begin{cases} \{0\} & \text{for } x \in (0, a_C] \\ [0, -\zeta x + \frac{a_C b}{a_C - a_I}] & \text{for } x \in (a_C, x_1] \\ [0, a_C] & \text{for } x \in (x_1, \infty) \end{cases}.$$

(ii) *If* $\delta = \mu_n$ *for* $n \in \mathbb{N}$, *then the optimal policy correspondence is given by*

$$h(x) = \begin{cases} \{0\} & \textit{for } x \in (0, a_C] \\ \{-\zeta x + \frac{a_C b}{a_C - a_I}\} & \textit{for } x \in (a_C, x_n] \\ [x_{n-1}, -\zeta x + \frac{a_C b}{a_C - a_I}] & \textit{for } x \in (x_n, x_{n+1}] \\ [x_{n-1}, x_n] & \textit{for } x \in (x_{n+1}, \infty) \end{cases}.$$

A.2 Proofs

Proof of Lemma 1 We first prove the "if" part. Let $\delta > 1/\theta$. Pick $\varepsilon > 0$ such that $\varepsilon < a_I$ and $\varepsilon < a_C$. Since $\delta > 1/\theta$, $\theta > 1/\delta$ and $\theta > (\theta + 1/\delta)/2 > 1/\delta$. Since $0 < \varepsilon < a_C$, $0 < \varepsilon < a_I$, and $(\theta + 1/\delta)/2 < \theta$, $(\varepsilon, (\theta + 1/\delta)\varepsilon/2) \in \Omega$ and $u(\varepsilon, (\theta + 1/\delta)\varepsilon/2) > 0 = u(0, 0)$. Moreover, since $\delta > 1/\theta$, $\delta((\theta + 1/\delta)\varepsilon/2) = (\delta\theta + 1)\varepsilon/2 > \varepsilon$. Thus, the economy is δ-normal. Now we turn to the "only if" part. Let $\delta \le 1/\theta$. For any $(x, x') \in \Omega$, $x' \le \theta x \le x/\delta$ which implies $x \ge \delta x'$. The equality holds only if $\delta = 1/\theta$ and $x' = \theta x$. However, if $x' = \theta x$, $u(x, x') = 0 = u(0, 0)$, so the economy is not δ-normal. Then, we have obtained the desired conclusion. □

Proof of Theorem 1 We first consider $0 < d < 1$. We adopt the standard "guess-and-verify" approach. Postulate a candidate value function based on the policy function $g(x) = (1 - d)x$ for any x:

$$W(x) = \begin{cases} \frac{x}{a_C(1-\delta(1-d))} & \text{for } x \in [0, a_C] \\ \frac{1-\delta^n}{1-\delta} + \frac{\delta^n(1-d)^n x}{a_C(1-\delta(1-d))} & \text{for } x \in (\frac{a_C}{(1-d)^{n-1}}, \frac{a_C}{(1-d)^n}] \end{cases}, \tag{A.1}$$

where $n = 1, 2, 3\ldots$ We now claim that for any x, $W(x)$ satisfies the Bellman equation

$$W(x) = \max_{x' \in \Gamma(x)} \{u(x, x') + \delta W(x')\}.$$

To this end, we consider four cases: (i) $x \in (0, a_I)$; (ii) $x \in [a_I, a_C]$; (iii) $x \in (a_C, \frac{a_C}{1-d}]$; (iv) $x \in (\frac{a_C}{(1-d)^{n-1}}, \frac{a_C}{(1-d)^n}]$ for $n = 2, 3, 4...$

Case (i) For any $(x, x') \in \Omega$ such that $x < a_I$, we have $(a_C - a_I)x' < ((1-d)(a_C - a_I) - b)x + a_C b$. Using the reduced-form utility function (3), for $x' \le a_C$, we have

$$\begin{aligned} W_0(x, x') \equiv u(x, x') + \delta W(x') &= \frac{a_I \theta}{a_C b} x - \frac{a_I}{a_C b} x' + \frac{\delta x'}{a_C(1 - \delta(1-d))}, \\ \frac{\partial W_0(x, x')}{\partial x'} = -\frac{a_I}{a_C b} + \frac{\delta}{a_C(1 - \delta(1-d))} &= \frac{a_I(\delta\theta - 1)}{a_C b(1 - \delta(1-d))} < 0, \end{aligned} \tag{A.2}$$

where the inequality follows from $\delta\theta < 1$. For $x' > a_C$, there exists a natural number n such that $x' \in (\frac{a_C}{(1-d)^{n-1}}, \frac{a_C}{(1-d)^n}]$. Similarly,

$$W_0(x,x') \equiv u(x,x') + \delta W(x') = \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' + \frac{\delta - \delta^{n+1}}{1-\delta} + \frac{\delta^{n+1}(1-d)^n x'}{a_C(1-\delta(1-d))},$$

$$\frac{\partial W_0(x,x')}{\partial x'} = -\frac{a_I}{a_C b} + \frac{\delta(\delta(1-d))^n}{a_C(1-\delta(1-d))} < -\frac{a_I}{a_C b} + \frac{\delta}{a_C(1-\delta(1-d))} = \frac{a_I(\delta\theta - 1)}{a_C b(1-\delta(1-d))} < 0,$$

where the first inequality follows from $\delta(1-d) < 1$ and $n \geq 1$ and the second inequality follows from $\delta\theta < 1$. Then, for $x' > a_C$, $W_0(x,x')$ strictly decreases with x'. Thus, for any x', $W_0(x,x')$ strictly decreases with x' and $W_0(x,x')$ attains its maximum when x' attains its minimum: $x' = (1-d)x$. Further, for $x \in (0, a_I)$,

$$W(x) = \frac{x}{a_C(1-\delta(1-d))} = \frac{x}{a_C} + \frac{\delta(1-d)x}{a_C(1-\delta(1-d))} = u(x,(1-d)x) + \delta W((1-d)x).$$

Then the Bellman equation is satisfied for Case (i).
Case (ii) For $(x,x') \in \Omega$ such that $x \in [a_I, a_C]$, there are two subcases: (a) $(a_C - a_I)x' \geq ((1-d)(a_C - a_I) - b)x + a_C b$ and (b) $(a_C - a_I)x' < ((1-d)(a_C - a_I) - b)x + a_C b$. We first consider Subcase (a). Using the reduced-form utility function (3), for $x' \leq a_C$,

$$W_0(x,x') \equiv u(x,x') + \delta W(x') = \frac{1-d}{b}x - \frac{1}{b}x' + 1 + \frac{\delta x'}{a_C(1-\delta(1-d))}.$$

$$\frac{\partial W_0(x,x')}{\partial x'} = -\frac{1}{b} + \frac{\delta}{a_C(1-\delta(1-d))} = \frac{\delta(b/a_C + (1-d)) - 1}{b(1-\delta(1-d))} < \frac{\delta\theta - 1}{b(1-\delta(1-d))} < 0, \tag{A.3}$$

where the first inequality follows from $a_C > a_I$ and $\theta = b/a_I + (1-d)$ and the second inequality follows from $\delta\theta < 1$. So $W_0(x,x')$ strictly decreases with x'. Similarly, we can show that $W_0(x,x')$ strictly decreases with x' for $x' > a_C$. For Subcase (b), similar to Case (i), we can show that $W_0(x,x')$ strictly decreases with x'. In sum, $W_0(x,x')$ attains its maximum when x' attains its minimum: $x' = (1-d)x$. Then, similar to Case (i), we can show that the Bellman equation is satisfied for Case (ii).
Case (iii) For any $(x,x') \in \Omega$ such that $x \in (a_C, \frac{a_C}{1-d}]$, we have $(a_C - a_I)x' \geq ((1-d)(a_C - a_I) - b)x + a_C b$. Similar to Subcase (a) of Case (ii), we can show that $(u(x,x') + \delta W(x'))$ attains its maximum when x' attains its minimum: $x' = (1-d)x$. Further, for $x \in (a_C, \frac{a_C}{1-d}]$,

$$W(x) = 1 + \frac{\delta(1-d)x}{a_C(1-\delta(1-d))} = u(x,(1-d)x) + \delta W((1-d)x),$$

where the last equation follows from $u(x,(1-d)x) = 1$ for $x \in (a_C, \frac{a_C}{1-d}]$ and $(1-d)x \leq a_C$. Then, the Bellman equation is satisfied for Case (iii).

Case (iv) For any $(x, x') \in \Omega$ such that $x \in (\frac{a_C}{(1-d)^{n-1}}, \frac{a_C}{(1-d)^n}]$ $(n = 2, 3, 4\ldots)$, we have $(a_C - a_I)x' \geq ((1-d)(a_C - a_I) - b)x + a_C b$, following again Subcase (a) of Case (ii), we can show that $(u(x, x') + \delta W(x'))$ attains its maximum when x' attains its minimum: $x' = (1-d)x$. Further, for $x \in (\frac{a_C}{(1-d)^{n-1}}, \frac{a_C}{(1-d)^n}]$ and any positive integer $n \geq 2$,

$$\begin{aligned} W(x) &= \frac{1-\delta^n}{1-\delta} + \frac{\delta^n(1-d)^n x}{a_C(1-\delta(1-d))} \\ &= 1 + \delta\left[\frac{1-\delta^{n-1}}{1-\delta} + \frac{\delta^{n-1}(1-d)^{n-1}(1-d)x}{a_C(1-\delta(1-d))}\right] \\ &= u(x,(1-d)x) + \delta W((1-d)x), \end{aligned}$$

where the last equation follows from $u(x,(1-d)x) = 1$ for $x \in (\frac{a_C}{(1-d)^{n-1}}, \frac{a_C}{(1-d)^n}]$ and $(1-d)x \in (\frac{a_C}{(1-d)^{n-2}}, \frac{a_C}{(1-d)^{n-1}}]$. Then the Bellman equation is satisfied for Case (iv).

In sum, we have verified that $W(x)$ is the value function satisfying the Bellman equation and the optimal policy is given by $g(x) = (1-d)x$ for any $x > 0$ and $0 < d < 1$. For the case of circulating capital ($d = 1$), we can apply essentially the same argument as above to show that the optimal policy is given by $g(x) = (1-d)x = 0$ for any $x > 0$ with the value function $V(x) = x/a_C$ for $x \leq a_C$ and $V(x) = 1$ for $x > a_C$. Thus, we have obtained the desired conclusion. □

Proof of Theorem 2 We first consider $0 < d < 1$. Following the proof of Theorem 1, we postulate a candidate value function $W(\cdot)$ to be the same as (A.1). We now claim that if $\delta < \mu_0$, then $W(\cdot)$ satisfies the Bellman equation. To this end, we consider two cases: (i) $x \leq a_C$; (ii) $x > a_C$. For Case (i), the proof follows entirely Case (i) in the proof of Theorem 1. For Case (ii), consider any $(x, x') \in \Omega$ such that $x > a_C$. There are two subcases: (a) $(a_C - a_I)x' \geq ((1-d)(a_C - a_I) - b)x + a_C b$ and (b) $(a_C - a_I)x' < ((1-d)(a_C - a_I) - b)x + a_C b$. We first consider Subcase (a). For $x' \leq a_C$,

$$W_0(x,x') \equiv u(x,x') + \delta W(x') = \frac{1-d}{b}x - \frac{1}{b}x' + 1 + \frac{\delta x'}{a_C(1-\delta(1-d))}.$$

$$\begin{aligned} \frac{\partial W_0(x,x')}{\partial x'} &= -\frac{1}{b} + \frac{\delta}{a_C(1-\delta(1-d))} = \frac{(b + a_C(1-d))\delta - a_C}{a_C b(1-\delta(1-d))} \\ &= \frac{b + a_C(1-d)}{a_C b(1-\delta(1-d))}(\delta - \mu_0) < 0, \end{aligned} \tag{A.4}$$

where the inequality follows from $\delta < \mu_0$. So $W_0(x, x')$ strictly decreases with x'. For $x' \in (\frac{a_C}{(1-d)^{n-1}}, \frac{a_C}{(1-d)^n}]$ with $n \in \mathbb{N}$,

$$\frac{\partial W_0(x,x')}{\partial x'} = -\frac{1}{b} + \frac{\delta^{n+1}(1-d)^n}{a_C(1-\delta(1-d))} < -\frac{1}{b} + \frac{\delta}{a_C(1-\delta(1-d))} < 0,$$

where the first inequality follows from $n \geq 1$ and $\delta(1-d) < 1$ and the second inequality follows from (A.4). This implies that $W_0(x, x')$ strictly decreases with x' for $x' > a_C$. Taken together, $W_0(x, x')$ strictly decreases with x' for any x' and it is maximized with $x' = (1-d)x$. For Subcase (b), similar to Case (i), we can show that $W_0(x, x')$ attains its maximum for $x' = (1-d)x$. Following the proof of Theorem 1, we can further obtain $W(x) = u(x, (1-d)x) + \delta W((1-d)x)$ for any $x > 0$. So we have verified that $W(\cdot)$ satisfies the Bellman equation. For the case of $d = 1$, we can follow essentially the same argument to show that $g(x) = 0$ for any $x > 0$. Thus, we have obtained the desired conclusion. □

Proof of Lemma 2 For $n = 0$, since $z_0(\cdot)$ is strictly increasing on $[0, 1/\theta]$ and by construction, $z_0(\mu_0) = 0$, we have $z_0(\delta) < z_0(\mu_0) = 0$ for $\delta \in [0, \mu_0)$ and $z_0(\delta) > z_0(\mu_0) = 0$ for $\delta \in (\mu_0, 1/\theta]$. Then, we just need to focus on $n \geq 1$ in this proof.

From Lemma A2, we know for $\delta \neq -1/\zeta$, $z_n(\delta) = 0$ if and only if $\tilde{z}_n(\delta) = 0$, where $\tilde{z}_n(\cdot)$ is defined in Eq. (A.9). We first investigate the property of $\tilde{z}_n(\cdot)$. From (c) of Lemma A2, there exists $\bar{\delta}$ such that $\tilde{z}'_n(\delta) > 0$ for $\delta \in [0, \bar{\delta})$ and $\tilde{z}'_n(\delta) < 0$ for $\delta \in (\bar{\delta}, 1/\theta]$, so $\tilde{z}_n(\cdot)$ is strictly increasing on $[0, \bar{\delta}]$ and strictly decreasing on $[\bar{\delta}, 1/\theta]$. To better explain our proof, we illustrate the qualitative features of $\tilde{z}_n$ in Fig. 11.

From (a.1)–(a.3) of Lemma A2, $\tilde{z}_n(-1/\zeta) = 0$, $\tilde{z}_n(0) < 0$, $\tilde{z}_n(1/\theta) < 0$, and $\tilde{z}''_n(\delta) < 0$ for $\delta \in [0, 1/\theta)$. There are three cases: (i) $\tilde{z}'_n(-1/\zeta) < 0$; (ii) $\tilde{z}'_n(-1/\zeta) > 0$; (iii) $\tilde{z}'_n(-1/\zeta) = 0$. For Case (i), given the monotonicity property of $\tilde{z}_n$, we must have $-1/\zeta > \bar{\delta}$. This case is illustrated in Panel (a) of Fig. 11. Since $-1/\zeta > \bar{\delta}$, $\tilde{z}'_n(\delta) < 0$ for $\delta \in [-1/\zeta, 1/\theta]$. From (a.1) of Lemma A2, $\tilde{z}_n(-1/\zeta) = 0$, and since $\tilde{z}'_n(\delta) < 0$ for $\delta \in [-1/\zeta, 1/\theta]$, $\tilde{z}_n(\delta) < 0$ for $\delta \in (-1/\zeta, 1/\theta]$. Since $\tilde{z}_n(\cdot)$ is strictly decreasing on $[\bar{\delta}, -1/\zeta]$ and $\tilde{z}_n(-1/\zeta) = 0$, $\tilde{z}_n(\delta) > 0$ for $\delta \in [\bar{\delta}, -1/\zeta)$. In particular, $\tilde{z}_n(\bar{\delta}) > 0$. Since $\tilde{z}_n(\cdot)$ is strictly increasing on $[0, \bar{\delta}]$ and $\tilde{z}_n(0) < 0$, by the continuity of $\tilde{z}_n(\cdot)$, there exists a unique root, denoted by μ_n, of $\tilde{z}_n(\delta) = 0$ on the interval $(0, \bar{\delta})$, and $\tilde{z}_n(\delta) > 0$ for $\delta \in (\mu_n, \bar{\delta})$. In sum, if $\tilde{z}'_n(-1/\zeta) < 0$, $\tilde{z}_n(\delta) = 0$ admits two roots, μ_n and $(-1/\zeta)$, in $[0, 1/\theta]$ such that $0 < \mu_n < -1/\zeta < 1/\theta$ and $\tilde{z}_n(\delta) > 0$ for $\delta \in (\mu_n, -1/\zeta)$. For Case (ii), $\tilde{z}'_n(-1/\zeta) > 0$, which is illustrated in Panel (b) of Fig. 11. Symmetrically, we can show that if $\tilde{z}'_n(-1/\zeta) > 0$, $\tilde{z}_n(\delta) = 0$ admits two roots, μ_n and $(-1/\zeta)$, in $[0, 1/\theta]$ such that $0 < -1/\zeta < \mu_n < 1/\theta$ and $\tilde{z}_n(\delta) > 0$ for $\delta \in (-1/\zeta, \mu_n)$. For Case (iii), $\tilde{z}'_n(-1/\zeta) = 0$, which is illustrated in Panel (c) of Fig. 11. In this case, $\bar{\delta} = -1/\zeta$, and thus, $\tilde{z}_n(\delta) = 0$ admits a unique root, $(-1/\zeta)$, and we let $\mu_n \equiv -1/\zeta$ in this case.

From (b) of Lemma A2, $\tilde{z}'_n(-1/\zeta) = 0$ if and only if $z_n(-1/\zeta) = 0$. From Lemma A2, we also know that if $\delta \neq -1/\zeta$, $z_n(\delta) = 0$ if and only if $\tilde{z}_n(\delta) = 0$. Thus, if $\tilde{z}'_n(-1/\zeta) \neq 0$, $z_n(-1/\zeta) \neq 0$, so μ_n must be the unique root of $z_n(\delta) = 0$ on $[0, 1/\theta]$. On the other hand, if $\tilde{z}'_n(-1/\zeta) = 0$, $z_n(-1/\zeta) = 0$, and $-1/\zeta(= \mu_n)$ is the unique root of $z_n(\delta) = 0$ on $[0, 1/\theta]$. Thus, $z_n(\delta) = 0$ on $[0, 1/\theta]$ always admits a unique root μ_n.

Next, we claim that $z_n(\delta) < 0$ for $\delta \in [0, \mu_n)$. By construction, we know $z_n(0) = -1/b < 0$. Suppose there exists $\hat{\delta} \in (0, \mu_n)$ such that $z_n(\hat{\delta}) \geq 0$. If $z_n(\hat{\delta}) = 0$, then it contradicts to μ_n being the unique root, so $z_n(\hat{\delta}) > 0$. Since $z_n(0) = -1/b < 0$, by the continuity of $z_n(\cdot)$ on $[0, 1/\theta]$, there exists a root in $(0, \hat{\delta})$ of the equation $z_n(\delta) = 0$. It again contradicts to μ_n being the unique root, thus establishing our claim.

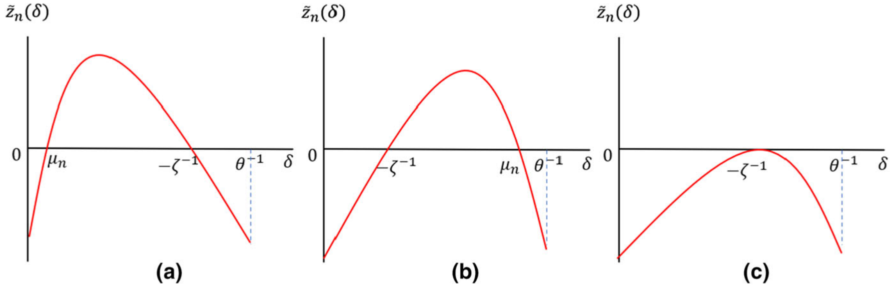

Fig. 11 Properties of $\tilde{z}_n$

Last, we claim that $z_n(\delta) > 0$ for $\delta \in (\mu_n, 1/\theta]$. There are two possible cases: (a) $\mu_n < -1/\zeta$ and (b) $\mu_n \geq -1/\zeta$. For (a), pick δ' in $(\mu_n, -1/\zeta)$. We have shown above that $\tilde{z}_n(\delta') > 0$ for $\delta' \in (\mu_n, -1/\zeta)$. Since $\delta' < -1/\zeta$ and $\zeta < 0$ (for $a_I > a_C$), $(1+\delta'\zeta) > 0$. Since $\tilde{z}_n(\delta') > 0$, $(1+\delta'\zeta) > 0$, $a_I > a_C$, $\delta' \neq -1/\zeta$, from Lemmas A1 and A2, we have

$$z_n(\delta') = \frac{\tilde{z}_n(\delta')}{a_C b(1-\delta(1-d))(a_I-a_C)(1+\delta'\zeta)} > 0.$$

Suppose on the contrary, there exists $\delta'' \in (\mu_n, 1/\theta]$ such that $z_n(\delta'') \leq 0$. If $z_n(\delta'') = 0$, it contradicts with μ_n being the unique root. If $z_n(\delta'') < 0$, since $z_n(\delta') > 0$, by the continuity of z_n on $[0, 1/\theta]$, there exists another root in $(\mu_n, 1/\theta)$, leading to a contradiction. Thus, we must have $z_n(\delta) > 0$ for $\delta \in (\mu_n, 1/\theta]$. For (b), since $\mu_n \geq -1/\zeta$, from the discussion above we know $\tilde{z}_n(\delta) < 0$ for any δ in $(\mu_n, 1/\theta]$. For $\delta > \mu_n \geq -1/\zeta$, $(1+\delta\zeta) < 0$ and since $\tilde{z}_n(\delta) < 0$, from Lemmas A1 and A2, we have $z_n(\delta) > 0$ for $\delta \in (\mu_n, 1/\theta]$, thus establishing the claim.

We have now obtained the desired conclusion. □

Proof of Lemma 3 From (A.8) in Lemma A1 stated below, if $\delta = -1/\zeta$, or equivalently, $-\delta\zeta = 1$, then

$$z_n(\delta) = \frac{-nba_C(1-d)\delta + nba_C - ba_I + 2ba_C}{a_C b(1-\delta(1-d))(a_I-a_C)\zeta}$$

Then, for $n > 1$, we have

$$z_n(\delta) - z_{n-1}(\delta) = \frac{ba_C(1-(1-d)\delta)}{a_C b(1-\delta(1-d))(a_I-a_C)\zeta} < 0,$$

where the inequality follows from $\delta(1-d) < 1$, $a_I > a_C$, and $\zeta < 0$. For $\delta \neq -1/\zeta$, from (A.8), we can write $z_n(\delta)$ as

$$z_n(\delta) = \frac{ba_I(-\zeta)^n(1-\theta\delta)\delta^{n+1} - a_C(a_I-a_C)(1-(1-d)\delta)^2}{a_C b(1-\delta(1-d))(a_I-a_C)(1+\delta\zeta)}.$$

Then, for $n > 1$, we have

$$
\begin{aligned}
z_n(\delta) - z_{n-1}(\delta) &= \frac{ba_I(-\zeta)^{n-1}(1-\theta\delta)\delta^n(-\delta\zeta - 1)}{a_C b(1-\delta(1-d))(a_I - a_C)(1+\delta\zeta)} \\
&= -\frac{ba_I(-\zeta)^{n-1}(1-\theta\delta)\delta^n}{a_C b(1-\delta(1-d))(a_I - a_C)} < 0,
\end{aligned}
$$

where the inequality follows from $-\zeta > 0$, $\delta < 1/\theta$, $\delta(1-d) < 1$, and $a_I > a_C$. Thus, we have shown that $z_n(\delta) < z_{n-1}(\delta)$ for any $n > 1$. Last,

$$
\begin{aligned}
z_1(\delta) - z_0(\delta) &= \delta\left(-\frac{1}{a_I - a_C} - \frac{\delta\zeta}{a_C(1-\delta(1-d))}\right) - \frac{\delta}{a_C(1-\delta(1-d))} \\
&= -\delta \cdot \frac{a_C(1-\delta(1-d)) + (1+\delta\zeta)(a_I - a_C)}{a_C(1-\delta(1-d))(a_I - a_C)} \\
&= -\delta \cdot \frac{a_I - a_I\delta(1-d) - b\delta}{a_C(1-\delta(1-d))(a_I - a_C)} \\
&= -\frac{\delta a_I(1-\delta\theta)}{a_C(1-\delta(1-d))(a_I - a_C)} < 0,
\end{aligned}
$$

where the inequality follows from $\delta\theta < 1$ and $a_I > a_C$. Thus, we have obtained the desired conclusion. □

Proof of Proposition 2 From Lemma 2, there is a unique root of $z_n(\delta) = 0$ on the interval $(0, 1/\theta)$. Denote this root by μ_n. Consider the sequence $\{\mu_n\}_{n=0}^{\infty}$. We now want to establish the monotonicity and the limit of this sequence. In particular, we want to show that the sequence $\{\mu_n\}_{n=0}^{\infty}$ satisfies (i) $\mu_n > \mu_{n-1}$ for any $n \in \mathbb{N}$ and (ii) $\lim_{n\to\infty} \mu_n = 1/\theta$.

To gain some intuition, we illustrate the determination of μ_n in Fig. 12. Let $v_n(\delta) \equiv ba_I(-\zeta)^n(1-\theta\delta)\delta^{n+1}$ and $w(\delta) \equiv a_C(a_I - a_C)(1-(1-d)\delta)^2$. Then, $\tilde{z}_n(\cdot)$, as defined in (A.9), can be written as $\tilde{z}_n(\delta) = v_n(\delta) - w(\delta)$, so $\tilde{z}(\delta) = 0$ if and only if $v_n(\delta) = w(\delta)$. It is straightforward to show that $w(\cdot)$ is strictly decreasing and $v_n(\cdot)$ is first strictly increasing and then strictly decreasing on $[0, 1/\theta]$. The two curves intersect with each other twice provided that $\mu_n \neq -1/\zeta$. One of the points of intersection always corresponds to $\delta = -1/\zeta$. The left panel shows how the curve of $v_n(\cdot)$ changes with n for $\mu_n < -1/\zeta$ while the right panel illustrates the case of $\mu_n > -1/\zeta$. As n increases, the red curve shifts to the right, thus leading to $\mu_{n+1} > \mu_n$.

To establish the monotonicity formally, from Lemma 2, we know for any $n \in \mathbb{N}$, $z_n(\delta) < 0$ for $\delta \in [0, \mu_n)$ and $z_n(\delta) > 0$ for $\delta \in (\mu_n, 1/\theta]$. From Lemma 3, $z_{n-1}(\delta) > z_n(\delta)$ for any $\delta \in (0, 1/\theta)$. In particular, $z_{n-1}(\delta) > z_n(\delta)$ for $\delta \in (\mu_n, 1/\theta)$, and by the continuity of $z_n(\cdot)$ and $z_{n-1}(\cdot)$, we must also have $z_{n-1}(1/\theta) \geq z_n(1/\theta)$. Then for $\delta \in (\mu_n, 1/\theta]$, $z_{n-1}(\delta) \geq z_n(\delta) > 0$. Further, $z_{n-1}(\mu_n) > z_n(\mu_n) = 0$. Thus, $z_{n-1}(\delta) > 0$ for $\delta \in [\mu_n, 1/\theta]$, so μ_{n-1}, defined as the unique root of the equation $z_{n-1}(\delta) = 0$ for $\delta \in [0, 1/\theta]$, has to be in $(0, \mu_n)$, which implies $\mu_{n-1} < \mu_n$.

We have now obtained the monotonic property of $\{\mu_n\}_{n=0}^{\infty}$. The next is to show that $\lim_{n\to\infty} \mu_n = 1/\theta$. We first note that, by construction, $\mu_n < 1/\theta$ for any $n \in \mathbb{N}$.

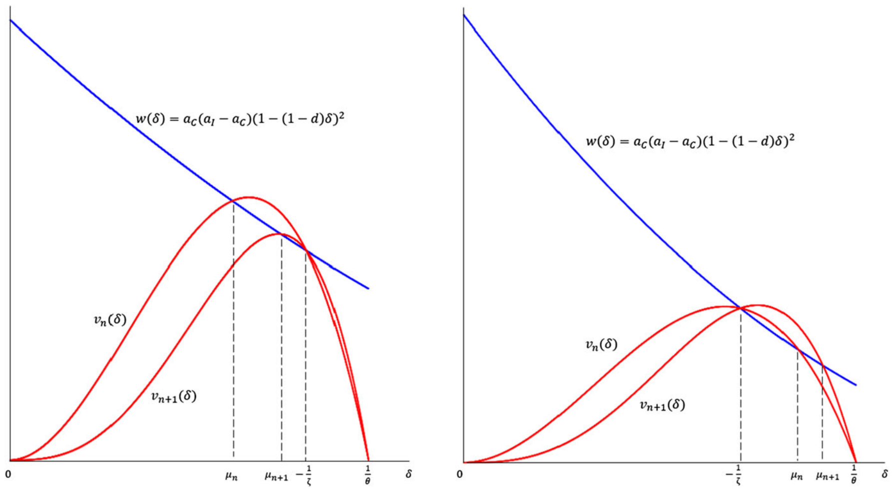

Fig. 12 Monotonicity of μ_n

Thus, the sequence $\{\mu_n\}_{n=0}^{\infty}$ is bounded above by $1/\theta$ and monotonic, so it must have a limit and $\lim_{n\to\infty}\mu_n \le 1/\theta$. Since we have

$$z_n\left(-\frac{1}{\zeta}\right) = -\frac{1}{b} - \frac{1}{\zeta}\cdot\left(-\frac{\sum_{i=0}^{n-1}(-(-1/\zeta)\zeta)^i}{a_I - a_C} + \frac{(-(-1/\zeta)\zeta)^n}{a_C(1+(1-d)/\zeta)}\right)$$
$$= -\frac{1}{b} - \frac{1}{a_C(\zeta+1-d)} + \frac{n}{\zeta(a_I - a_C)},$$

$z_n(-1/\zeta)$ is linear in n. Since $\zeta < 0$ (for $a_I > a_C$), $z_n(-1/\zeta)$ is strictly decreasing in n. Thus, there exists $n_0 \in \mathbb{N}$ such that for any $n > n_0$, $z_n(-1/\zeta) < 0$, and by Lemma 2, it implies $\mu_n > -1/\zeta$. From Lemma A2, for any $n > n_0$, we have

$$\begin{aligned}
\tilde{z}_n(\mu_n) &= ba_I(-\zeta)^n(1-\theta\mu_n)\mu_n^{n+1} - a_C(a_I - a_C)(1-(1-d)\mu_n)^2 = 0 \\
&\Leftrightarrow \lim_{n\to\infty} ba_I(-\zeta)^n(1-\theta\mu_n)\mu_n^{n+1} - a_C(a_I - a_C)(1-(1-d)\mu_n)^2 = 0 \\
&\Leftrightarrow \lim_{n\to\infty} ba_I(-\zeta)^n(1-\theta\mu_n)\mu_n^{n+1} = \lim_{n\to\infty} a_C(a_I - a_C)(1-(1-d)\mu_n)^2 < \infty,
\end{aligned}$$

where the second line follows from $\tilde{z}_n(\mu_n) = 0$ for any $n > n_0$ and the third line follows from the fact that $\{\mu_n\}_{n=0}^{\infty}$ has a limit and $\lim_{n\to\infty} \le 1/\theta$. Suppose $\lim_{n\to\infty}\mu_n < 1/\theta$. Since $\mu_n > -1/\zeta$ for $n > n_0$ and $\{\mu_n\}_{n=0}^{\infty}$ is strictly increasing, then

$$\lim_{n\to\infty} ba_I(-\zeta)^n(1-\theta\mu_n)\mu_n^{n+1} \ge \lim_{n\to\infty} ba_I(1-\theta\lim_{n\to\infty}\mu_n)\cdot(-\zeta)^n\mu_{n_0+1}^{n+1} = \infty,$$

where the last equality follows from $\lim_{n\to\infty}\mu_n < 1/\theta$ and $\mu_{n_0+1} > -1/\zeta$, leading to a contradiction. Thus, we must have $\lim_{n\to\infty}\mu_n = 1/\theta$.

We have now obtained the desired conclusion. □

Proof of Lemma 4 We first consider the case of $\zeta \neq -1$. Since $\zeta \neq -1$, $b/(a_C - a_I) + d \neq 0$ or equivalently, $b + (a_C - a_I)d \neq 0$. Further, we have

$$\frac{a_C b}{b + d(a_C - a_I)} = \frac{a_C(\zeta + 1 - d)}{\zeta + 1} \quad \text{and} \quad \frac{a_C b}{a_C - a_I} = a_C(\zeta + 1 - d).$$

Since by construction, $x_n = -(x_{n-1} - a_C b/(a_C - a_I))/\zeta$, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} x_n - \frac{a_C b}{b + d(a_C - a_I)} &= -\frac{1}{\zeta}\left(x_{n-1} - \frac{a_C b}{a_C - a_I}\right) - \frac{a_C b}{b + d(a_C - a_I)} \\ &= -\frac{1}{\zeta}\left(x_{n-1} - a_C(\zeta + 1 - d)\right) - \frac{a_C(\zeta + 1 - d)}{\zeta + 1} \\ &= -\frac{1}{\zeta}\left(x_{n-1} - \frac{a_C(\zeta + 1 - d)}{\zeta + 1}\right) \\ &= -\frac{1}{\zeta}\left(x_{n-1} - \frac{a_C b}{b + d(a_C - a_I)}\right) \\ &= \frac{1}{(-\zeta)^n}\left(x_0 - \frac{a_C b}{b + d(a_C - a_I)}\right) \\ &= -\frac{d a_C(a_I - a_C)}{(b + d(a_C - a_I))(-\zeta)^n}, \end{aligned} \tag{A.5}$$

where the last equality follows from $x_0 = a_C$. Since $a_I > a_C$, $\zeta < 0$, and we consider $\zeta \neq -1$, so there are two cases: (i) $\zeta < -1$ and (ii) $0 > \zeta > -1$. For (i), since $\zeta < -1$, $(1 + \zeta) < 0$. Since $a_I > a_C$ and $\zeta < -1$, $b + d(a_C - a_I) > 0$. Thus, $(1 + \zeta)/(b + d(a_C - a_I)) < 0$. For (ii), since $\zeta > -1$, $1 + \zeta > 0$. Since $\zeta > -1$ and $a_I > a_C$, $b + d(a_C - a_I) < 0$. Again, we have $(1 + \zeta)/(b + d(a_C - a_I)) < 0$. For both cases, we then have

$$\begin{aligned} x_n - x_{n-1} &= (x_n - \frac{a_C b}{b + d(a_C - a_I)}) - (x_{n-1} - \frac{a_C b}{b + d(a_C - a_I)}) \\ &= -\frac{d a_C(a_I - a_C)}{(b + d(a_C - a_I))(-\zeta)^n} + \frac{d a_C(a_I - a_C)}{(b + d(a_C - a_I))(-\zeta)^{n-1}} \\ &= -\frac{d a_C(a_I - a_C)(1 + \zeta)}{(b + d(a_C - a_I))(-\zeta)^n} > 0, \end{aligned}$$

where the inequality follows from $a_I > a_C$, $\zeta < 0$ and $(1 + \zeta)/(b + d(a_C - a_I)) < 0$ for both cases. Thus, $x_n > x_{n-1}$ for any $n \in \mathbb{N}$ and $\zeta \neq -1$. For $\zeta = -1$, $x_n = x_{n-1} + a_C b/(a_I - a_C)$. Since $a_I > a_C$, x_n strictly increases with n.

Moreover, since $a_I > a_C > 0$, $-\zeta > \theta$. Then, for $\theta \geq 1$, $\zeta < -\theta \leq -1$, and from (A.5), this implies $\lim_{n\to\infty} x_n = \frac{a_C b}{b + d(a_C - a_I)}$. For $\theta > 1$, $\hat{x} = \frac{a_C b}{b + d(a_C - a_I)}$, so we have $\lim_{n\to\infty} x_n = \hat{x}$. For $\theta = 1$, $b/a_I = d$, and thus, $\lim_{n\to\infty} x_n = a_I$. We have then obtained the desired conclusion. □

Proof of Proposition 3 We adopt the standard guess-and-verify approach. Let $\mu_{n-1} < \delta < \mu_n$ for some $n \in \mathbb{N}$. Consider the following candidate policy function

$$\bar{g}(x) = \begin{cases} (1-d)x & \text{for } x \in (0, a_C] \\ -\zeta x + \frac{a_C b}{a_C - a_I} & \text{for } x \in (a_C, x_n] \\ x_{n-1} & \text{for } x \in (x_n, \frac{x_{n-1}}{1-d}] \\ (1-d)x & \text{for } x \in (\frac{x_{n-1}}{1-d}, \infty) \end{cases},$$

where $x_0 = a_C$ and from Lemma A3, $x_n = \hat{x} - (\hat{x} - a_C)/(-\zeta)^n$ (for $\theta = 1$, let $\hat{x} = a_C b/(b + d(a_C - a_I)) = a_I$). To see that $\bar{g}(\cdot)$ is well defined, we need to verify (a) $a_C < x_n$, (b) $x_n < x_{n-1}/(1-d)$, and (c) $(x, \bar{g}(x)) \in \Omega$ for any $x > 0$. Since $a_C < a_I$, from Lemma 4, $x_n > x_{n-1}$. Since $x_0 = a_C$, $x_n > x_0 = a_C$, for any $n \in \mathbb{N}$. Then, (a) is verified. Since $x_n > a_C$ and $a_C < a_I$, $(\zeta + 1 - d)x_n = bx_n/(a_C - a_I) < a_C b/(a_C - a_I)$, or equivalently, $x_n < (a_C b/(a_C - a_I) - \zeta x_n)/(1-d) = x_{n-1}/(1-d)$, where the equality follows from the construction of the sequence $\{x_n\}_{n=0}^{\infty}$. Then, (b) is verified. For $x \in (0, a_C] \cup (x_{n-1}/(1-d), \infty)$, $(x, (1-d)x) \in \Omega$. Since $a_C < a_I$ and $\theta \geq 1$, $\zeta < -\theta \leq -1$. Since $\zeta < -1$, $x_n = \hat{x} - (\hat{x} - a_C)/(-\zeta)^n$, and $\hat{x} > a_C$ (from $a_C < a_I$ and $\theta \geq 1$), we have $x_n < \hat{x}$. We have shown $x_n > a_C$, and from $a_C < a_I$ and $\theta \geq 1$, $\hat{x} \leq a_I$, so $a_C < x_n < \hat{x} \leq a_I$. Since $a_C < x_n < a_I$, $(x, -\zeta x + a_C b/(a_C - a_I)) = (x, \hat{x} + \zeta(\hat{x} - x)) \in \Omega$ for any $x \in (a_C, x_n]$. For $x \in (x_n, x_{n-1}/(1-d)]$, $x \leq x_{n-1}/(1-d)$. Then, $(1-d)x \leq x_{n-1}$. Since $\theta \geq 1$ and $x_{n-1} < x_n$, $\theta x \geq x > x_n > x_{n-1}$. Further, if $x > a_I$, $(1-d)x + b \geq (1-d)a_I + b = \theta a_I \geq a_I > x_n > x_{n-1}$. Since $\theta x \leq (1-d)x + b$ if and only if $x \leq a_I$, we have shown that $(1-d)x \leq x_{n-1} < \min\{\theta x, (1-d)x + b\}$ for any $x \in (x_n, x_{n-1}/(1-d)]$. Then, (c) is verified. Based on the policy function $\bar{g}(\cdot)$, we postulate the following value function

$$W(x) = \begin{cases} \frac{x}{a_C(1-\delta(1-d))} & \text{for } x \in [0, a_C] \\ \left(-\frac{\sum_{i=0}^{m-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^m}{a_C(1-\delta(1-d))}\right)(x - \hat{x}) & \text{for } x \in (x_{m-1}, x_m], \\ + \frac{(1-\delta^m)(a_I - \hat{x})}{(1-\delta)(a_I - a_C)} + \frac{\delta^m \hat{x}}{a_C(1-\delta(1-d))} & \quad m = 1, 2, \ldots, n \\ \frac{1-d}{b}x + \frac{b - x_{n-1}}{b} + \frac{\sum_{i=1}^{n}\delta^i(a_I - x_{n-i})}{a_I - a_C} & \\ \quad + \frac{\delta^{n+1}(1-d)a_C}{1-\delta(1-d)} & \text{for } x \in (x_n, \frac{x_{n-1}}{1-d}] \\ \delta^{\ell}\left(-\frac{\sum_{i=0}^{n-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^n}{a_C(1-\delta(1-d))}\right) & \\ \quad ((1-d)^{\ell}x - \hat{x}) & \text{for } x \in \left(\frac{x_{n-1}}{(1-d)^{\ell}}, \frac{x_n}{(1-d)^{\ell}}\right] \\ + \frac{1-\delta^{\ell}}{1-\delta} + \delta^{\ell}\left(\frac{(1-\delta^n)(a_I - \hat{x})}{(1-\delta)(a_I - a_C)} + \frac{\delta^n \hat{x}}{a_C(1-\delta(1-d))}\right) & \quad \ell = 1, 2, \ldots \\ \frac{\delta^{\ell}(1-d)^{\ell+1}}{b}x + \frac{1-\delta^{\ell}}{1-\delta} + \delta^{\ell}\left(\frac{b - x_{n-1}}{b}\right. & \text{for } x \in (\frac{x_n}{(1-d)^{\ell}}, \frac{x_{n-1}}{(1-d)^{\ell+1}}] \\ \left. + \frac{\sum_{i=1}^{n}\delta^i(a_I - x_{n-i})}{a_I - a_C} + \frac{\delta^{n+1}(1-d)a_C}{1-\delta(1-d)}\right) & \quad \ell = 1, 2, \ldots \end{cases}$$

Before we verify that $W(\cdot)$ satisfies the value function, we first show how we obtain the postulated value function. For $x \in (0, a_C]$,

$$W(x) = \sum_{i=0}^{\infty} \delta^i u((1-d)^i x, (1-d)^{i+1} x) = \sum_{i=0}^{\infty} \delta^i \frac{(1-d)^i x}{a_C} = \frac{x}{a_C(1-\delta(1-d))}.$$

For $x \in (x_{m-1}, x_m]$ with $m \in \{1, 2, \ldots, n\}$, let $\bar{f}(x) = -\zeta x + a_C b/(a_C - a_I) = \zeta(\hat{x} - x) + \hat{x}$. Then, we have

$$\begin{aligned}
W(x) &= \sum_{i=0}^{m-1} \delta^i u(\bar{f}^i(x), \bar{f}^{i+1}(x)) + \delta^m W(\bar{f}^m(x)) \\
&= \sum_{i=0}^{m-1} \delta^i \frac{a_I - \bar{f}^i(x)}{a_I - a_C} + \frac{\delta^m \bar{f}^m(x)}{a_C(1-\delta(1-d))} \\
&= \sum_{i=0}^{m-1} \delta^i \frac{a_I - \hat{x} + (-\zeta)^i(\hat{x} - x)}{a_I - a_C} + \frac{\delta^m[\hat{x} - (-\zeta)^m(\hat{x} - x)]}{a_C(1-\delta(1-d))} \\
&= \left(-\frac{\sum_{i=0}^{m-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^m}{a_C(1-\delta(1-d))} \right)(x - \hat{x}) \\
&\quad + \frac{(1-\delta^m)(a_I - \hat{x})}{(1-\delta)(a_I - a_C)} + \frac{\delta^m \hat{x}}{a_C(1-\delta(1-d))}
\end{aligned}$$

where the first equation follows from $\bar{g}^i(x) = \bar{f}^i(x) \in (x_{m-i-1}, x_{m-i}]$ for $i = 0, 1, \ldots, m-1$, the second equation follows from

$$\begin{aligned}
u(\bar{f}^i(x), \bar{f}^{i+1}(x)) &= u(\bar{f}^i(x), \zeta(\hat{x} - \bar{f}^i(x)) + \hat{x}) \\
&= \frac{(1-d)}{b} \bar{f}^i(x) - \frac{\zeta(\hat{x} - \bar{f}^i(x)) + \hat{x}}{b} + 1 = \frac{(a_I - \bar{f}^i(x))}{(a_I - a_C)},
\end{aligned}$$

and the third equation follows from $\bar{f}^i(x) = \hat{x} - (-\zeta)^i(\hat{x} - x)$. It should be noted that $-\zeta > \theta$, so we cannot rule out the possibility of $(-\delta\zeta) = 1$.

For $x \in (x_n, x_{n-1}/(1-d)]$, since $(1-d)x_0 = (1-d)a_C = -\zeta a_C + a_C b/(a_C - a_I) = -\zeta x_0 + a_C b/(a_C - a_I)$, $u(a_C, (1-d)a_C) = u(\bar{f}^{n-1}(x_{n-1}), (1-d)\bar{f}^{n-1}(x_{n-1})) = u(\bar{f}^{n-1}(x_{n-1}), \bar{f}^n(x_{n-1}))$, and we have

$$\begin{aligned}
W(x) &= u(x, x_{n-1}) + \sum_{i=1}^{n} \delta^i u(\bar{f}^{i-1}(x_{n-1}), \bar{f}^i(x_{n-1})) + \delta^{n+1} W(\bar{f}^n(x_{n-1})) \\
&= \frac{1-d}{b} x - \frac{x_{n-1}}{b} + 1 + \sum_{i=1}^{n} \delta^i \frac{a_I - \bar{f}^{i-1}(x_{n-1})}{a_I - a_C} + \frac{\delta^{n+1} \bar{f}^n(x_{n-1})}{a_C(1-\delta(1-d))} \\
&= \frac{1-d}{b} x + \frac{b - x_{n-1}}{b} + \frac{\sum_{i=1}^{n} \delta^i (a_I - x_{n-i})}{a_I - a_C} + \frac{\delta^{n+1}(1-d)a_C}{1-\delta(1-d)}
\end{aligned}$$

where the first equation follows from $\bar{g}^i(x) = \bar{f}^{i-1}(x_{n-1}) = x_{n-i}$ for $i = 1, \dots, n$; the second equation follows from $x_{n-1} = \zeta(\hat{x} - x_n) + \hat{x} < \zeta(\hat{x} - x) + \hat{x}$ for $x \in (x_n, x_{n-1}/(1-d)]$, $u(\bar{f}^{i-1}(x_{n-1}), \bar{f}^i(x_{n-1})) = u(\bar{f}^{i-1}(x_{n-1}), \bar{f}(\bar{f}^{i-1}(x_{n-1}))) = (a_I - \bar{f}^{i-1}(x_{n-1}))/(a_I - a_C)$, and $\bar{f}^n(x_{n-1}) = (1-d)a_C < a_C$; the third equation follows from $\bar{f}^{i-1}(x_{n-1}) = x_{n-i}$ and $\bar{f}^n(x_{n-1}) = \bar{f}(x_0) = (1-d)a_C$.

For $x \in \left(\frac{x_{n-1}}{(1-d)^\ell}, \frac{x_n}{(1-d)^\ell}\right]$ with $\ell \in \mathbb{N}$, we have

$$
\begin{aligned}
W(x) &= \sum_{i=0}^{\ell-1} \delta^i + \delta^\ell W((1-d)^\ell x) \\
&= \frac{1-\delta^\ell}{1-\delta} + \delta^\ell \left[\left(-\frac{\sum_{i=0}^{n-1} (-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^n}{a_C(1-\delta(1-d))} \right) ((1-d)^\ell x - \hat{x}) \right. \\
&\quad \left. + \frac{(1-\delta^n)(a_I - \hat{x})}{(1-\delta)(a_I - a_C)} + \frac{\delta^n \hat{x}}{a_C(1-\delta(1-d))} \right] \\
&= \delta^\ell \left(-\frac{\sum_{i=0}^{n-1} (-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^n}{a_C(1-\delta(1-d))} \right) ((1-d)^\ell x - \hat{x}) \\
&\quad + \frac{1-\delta^\ell}{1-\delta} + \delta^\ell \left(\frac{(1-\delta^n)(a_I - \hat{x})}{(1-\delta)(a_I - a_C)} + \frac{\delta^n \hat{x}}{a_C(1-\delta(1-d))} \right),
\end{aligned}
$$

where the first equation follows from $\bar{g}^i(x) > x_{n-1}/(1-d)$ for $i = 0, 1, \dots, \ell-1$ and the second equation follows from $(1-d)^\ell x \in (x_{n-1}, x_n]$.

For $x \in (\frac{x_n}{(1-d)^\ell}, \frac{x_{n-1}}{(1-d)^{\ell+1}}]$ with $\ell \in \mathbb{N}$,

$$
\begin{aligned}
W(x) &= \sum_{i=0}^{\ell-1} \delta^i + \delta^\ell W((1-d)^\ell x) \\
&= \frac{1-\delta^\ell}{1-\delta} + \delta^\ell \left[\frac{1-d}{b}(1-d)^\ell x + \frac{b - x_{n-1}}{b} + \frac{\sum_{i=1}^n \delta^i (a_I - x_{n-i})}{a_I - a_C} \right. \\
&\quad \left. + \frac{\delta^{n+1}(1-d)a_C}{1-\delta(1-d)} \right] \\
&= \frac{\delta^\ell (1-d)^{\ell+1}}{b} x + \frac{1-\delta^\ell}{1-\delta} \\
&\quad + \delta^\ell \left(\frac{b - x_{n-1}}{b} + \frac{\sum_{i=1}^n \delta^i (a_I - x_{n-i})}{a_I - a_C} + \frac{\delta^{n+1}(1-d)a_C}{1-\delta(1-d)} \right),
\end{aligned}
$$

where the first inequality follows from $\bar{g}^i(x) > x_{n-1}/(1-d)$ for $i = 0, 1, \dots, \ell-1$ and the second inequality follows from $(1-d)^\ell x \in (x_n, x_{n-1}/(1-d)]$.

We now turn to the verification of whether $W(\cdot)$ satisfies the Bellman equation. For $x \in (0, a_C]$, $(x, x') \in \Omega$ implies that $x' \geq \zeta(\hat{x} - x) + \hat{x}$. Using the reduced-form utility function (3), we have

$$W_0(x,x') \equiv u(x,x') + \delta W(x') = \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' + \delta W(x').$$

For $x' \in (0, a_C]$,

$$\frac{\partial W_0(x,x')}{\partial x'} = -\frac{a_I}{a_C b} + \frac{\delta}{a_C(1-\delta(1-d))} = \frac{a_I(\delta\theta - 1)}{a_C b(1-\delta(1-d))} < 0,$$

where the inequality follows from $\delta\theta < 1$. For $x' \in (x_{m-1}, x_m]$ for some $m \in \{1,2,\ldots,n\}$,

$$\begin{aligned}
\frac{\partial W_0(x,x')}{\partial x'} &= -\frac{a_I}{a_C b} + \delta\left(-\frac{\sum_{i=0}^{m-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^m}{a_C(1-\delta(1-d))}\right)\\
&= -\frac{a_I}{a_C b} + \delta\left(\frac{1}{b} + z_m(\delta)\right) \leq -\frac{a_I}{a_C b} + \delta\left(\frac{1}{b} + z_1(\delta)\right)\\
&= -\frac{a_I}{a_C b} + \delta\left(-\frac{1}{a_I - a_C} - \frac{\zeta\delta}{a_C(1-\delta(1-d))}\right)\\
&< -\frac{a_I}{a_C b} + \frac{\delta}{a_C(1-\delta(1-d))} = \frac{a_I(\delta\theta-1)}{a_C b(1-\delta(1-d))} < 0,
\end{aligned}$$

where the first inequality follows from Lemma 3, the second inequality follows from $\delta\theta < 1$ and Lemma A4, and the third inequality follows from $\delta\theta < 1$. For $x' \in (x_n, x_{n-1}/(1-d)]$, we have

$$\frac{\partial W_0(x,x')}{\partial x'} = -\frac{a_I}{a_C b} + \frac{(1-d)\delta}{b} < 0,$$

where the inequality follows from $a_I > a_C$ and $(1-d)\delta < 1$. For $x' \in \left(\frac{x_{n-1}}{(1-d)^\ell}, \frac{x_n}{(1-d)^\ell}\right]$ with $\ell \in \mathbb{N}$, then

$$\begin{aligned}
\frac{\partial W_0(x,x')}{\partial x'} &= -\frac{a_I}{a_C b} + \delta^{\ell+1}(1-d)^\ell\left(-\frac{\sum_{i=0}^{n-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^n}{a_C(1-\delta(1-d))}\right),\\
&\leq -\frac{a_I}{a_C b} + \delta^{\ell+1}(1-d)^\ell\left(-\frac{1}{a_I - a_C} - \frac{\zeta\delta}{a_C(1-\delta(1-d))}\right)\\
&\leq -\frac{a_I}{a_C b} + \delta\left(-\frac{1}{a_I - a_C} - \frac{\zeta\delta}{a_C(1-\delta(1-d))}\right) < 0,
\end{aligned}$$

where the first inequality follows from Lemma 3, the second inequality follows from $\delta(1-d) < 1$, and third inequality follows from $\delta\theta < 1$ and Lemma A4. If $x' \in$

$(\frac{x_n}{(1-d)^\ell}, \frac{x_{n-1}}{(1-d)^{\ell+1}}]$ with $\ell \in \mathbb{N}$, then

$$\frac{\partial W_0(x, x')}{\partial x'} = -\frac{a_I}{a_C b} + \frac{(1-d)^{\ell+1}\delta^{\ell+1}}{b} < 0,$$

where the inequality follows from $a_I > a_C$ and $(1-d)\delta < 1$. Thus, for any x', $W_0(x, x')$ strictly decreases with x', so $W_0(x, x')$ attains its maximum when x' attains its minimum: $x' = (1-d)x$.

In what follows, for any $(x, x') \in \Omega$ such that $x' \geq \zeta(\hat{x} - x) + \hat{x}$, following the same argument as in the case of $x \leq a_C$ above, we can show that $W_0(x, x')$ strictly decreases with x', so $W_0(x, x')$ attains its maximum only if $x' \leq \zeta(\hat{x} - x) + \hat{x}$. We thus focus on $x' \geq \zeta(\hat{x} - x) + \hat{x}$ for $x > a_C$. Using the reduced-form utility function, we have

$$W_0(x, x') \equiv u(x, x') + \delta W(x') = \frac{1-d}{b}x - \frac{1}{b}x' + 1 + \delta W(x').$$

Consider $x \in (x_{m-1}, x_m]$ with $m \in \{1, 2, \ldots, n\}$. Since $\zeta < 0$, $x' \leq \zeta(\hat{x}-x)+\hat{x} \leq \zeta(\hat{x} - x_m) + \hat{x} = x_{m-1}$, where the last equality follows from the construction of x_m. If $x' < a_C$, then

$$\frac{\partial W_0(x, x')}{\partial x'} = -\frac{1}{b} + \frac{\delta}{a_C(1-\delta(1-d))} > 0,$$

where the inequality follows from $(a_C(1-d)+b)\delta > a_C$, or equivalently, $\delta > \mu_0 = 1/(b/a_C + (1-d))$. If $x' \in (x_{m'-1}, x_{m'}]$ for some $m' \in \mathbb{N}$ and $m' \leq m-1$, then

$$\frac{\partial W_0(x, x')}{\partial x'} = -\frac{1}{b} + \delta\left(-\frac{\sum_{i=0}^{m'-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^{m'}}{a_C(1-\delta(1-d))}\right) = z_{m'}(\delta) > 0,$$

where the inequality follows from $m' \leq m-1 \leq n-1$, $\delta > \mu_{n-1} \geq \mu_{m'}$ (from Lemma 2), and $z_{m'}(\delta) > 0$ for $\delta > \mu_{m'}$ (from Lemma 2). Thus, for any $x' \leq \zeta(\hat{x} - x) + \hat{x}$, $W_0(x, x')$ strictly increases with x' and for any $x' \geq \zeta(\hat{x} - x) + \hat{x}$, $W_0(x, x')$ strictly decreases with x', so $W_0(x, x')$ attains its maximum when $x' = \zeta(\hat{x} - x) + \hat{x}$.

Consider $x \in (x_n, x_{n-1}/(1-d)]$. For $x' \in (0, a_C]$, following the argument for $x \in (x_{m-1}, x_m]$ with $m \in \{1, 2, \ldots, n\}$, we can show that $W_0(x, x')$ strictly increases with x'. For $x' \in (x_{m-1}, x_m]$ for some $m \in \{1, 2, \ldots, n\}$,

$$\frac{\partial W_0(x, x')}{\partial x'} = -\frac{1}{b} + \delta\left(-\frac{\sum_{i=0}^{m-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^{m}}{a_C(1-\delta(1-d))}\right) = z_m(\delta). \tag{A.6}$$

Since $\mu_{n-1} < \delta < \mu_n$, from Lemma 2, $\mu_m \leq \mu_{n-1} < \delta$ for $m = 1, 2, \ldots, n-1$, and $\mu_m = \mu_n > \delta$ for $m = n$. From Lemma 2, $z_m(\delta) > 0$ for $m = 1, 2, \ldots, n-1$, and

$z_m(\delta) < 0$ for $m = n$. Thus, $W_0(x, x')$ strictly increases with x' for $x' \in (x_0, x_{n-1}]$ and strictly decreases with x' for $x' \in (x_{n-1}, x_n]$. For $x' \in (x_n, x_{n-1}/(1-d)]$,

$$\frac{\partial W_0(x, x')}{\partial x'} = -\frac{1}{b} + \frac{\delta(1-d)}{b} < 0,$$

which follows from $\delta(1-d) < 1$. For $x' \in \left(\frac{x_{n-1}}{(1-d)^\ell}, \frac{x_n}{(1-d)^\ell}\right]$ for $\ell \in \mathbb{N}$, then

$$\begin{aligned}\frac{\partial W_0(x, x')}{\partial x'} &= -\frac{1}{b} + \delta^{\ell+1}(1-d)^\ell \left(-\frac{\sum_{i=0}^{n-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^n}{a_C(1-\delta(1-d))}\right) \\ &= -\frac{1}{b} + \delta^{\ell+1}(1-d)^\ell \left(z_n(\delta) + \frac{1}{b}\right) < -\frac{1}{b} + \frac{\delta^{\ell+1}(1-d)^\ell}{b} < 0,\end{aligned}$$

where the first inequality follows from $\delta < \mu_n$ and $z_n(\delta) < 0$ (from Lemma 2) and the second inequality follows from $\delta(1-d) < 1$. For $x' \in (\frac{x_n}{(1-d)^\ell}, \frac{x_{n-1}}{(1-d)^{\ell+1}}]$ for $\ell \in \mathbb{N}$, again, we have

$$\frac{\partial W_0(x, x')}{\partial x'} = -\frac{1}{b} + \frac{\delta^{\ell+1}(1-d)^\ell}{b} < 0.$$

Thus, we have shown that $W_0(x, x')$ strictly increases with x' for $x' < x_{n-1}$ and strictly decreases with x' for $x' > x_{n-1}$, so $W_0(x, x')$ attains its maximum when $x' = x_{n-1}$.

Consider $x > x_{n-1}/(1-d)$ with $x' \le \zeta(\hat{x} - x) + \hat{x}$. Since $x' \ge (1-d)x > x_{n-1}$, following the argument for $x \in (x_n, \frac{x_{n-1}}{1-d}]$, we can show that $W_0(x, x')$ strictly decreases with x' for $x' > x_{n-1}$. Thus, $W_0(x, x')$ attains its maximum when $x' = (1-d)x$.

We have now shown that for every $x > 0$, $W_0(x, x') = u(x, x') + \delta W(x')$ is maximized for $x' = \bar{g}(x)$. Since $W(\cdot)$ is constructed from the policy function $\bar{g}(\cdot)$, we have $W(x) = u(x, \bar{g}(x)) + \delta W(\bar{g}(x))$ for every $x > 0$. So $W(\cdot)$ satisfies the Bellman equation and $\bar{g}(x)$ is the corresponding optimal policy. Thus, we have obtained the desired conclusion. □

Proof of Proposition 4 Let $d = 1$. Let $\mu_{n-1} < \delta < \mu_n$ for some $n \in \mathbb{N}$. We postulate the following value function

$$W(x) = \begin{cases} \frac{x}{a_C} & \text{for } x \in [0, a_C] \\ \left(-\frac{\sum_{i=0}^{m-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^m}{a_C}\right)(x - \hat{x}) & \text{for } x \in (x_{m-1}, x_m], \\ +\frac{(1-\delta^m)(a_I - \hat{x})}{(1-\delta)(a_I - a_C)} + \frac{\delta^m \hat{x}}{a_C} & \quad m = 1, 2, \ldots, n \\ \frac{b - x_{n-1}}{b} + \frac{\sum_{i=1}^n \delta^i (a_I - x_{n-i})}{a_I - a_C} & \text{for } x \in (x_n, \infty) \end{cases}.$$

The verification of $W(\cdot)$ satisfying the Bellman equation follows closely the proof of Proposition 3. For $x \in (0, a_C]$, $(x, x') \in \Omega$ implies that $x' \ge \zeta(\hat{x} - x) + \hat{x}$. Using the reduced-form utility function, we have

$$W_0(x, x') \equiv u(x, x') + \delta W(x') = \frac{a_I \theta}{a_C b} x - \frac{a_I}{a_C b} x' + \delta W(x').$$

For $x' \in (0, a_C]$,

$$\frac{\partial W_0(x, x')}{\partial x'} = -\frac{a_I}{a_C b} + \frac{\delta}{a_C} = \frac{a_I(\delta\theta - 1)}{a_C b} < 0,$$

where the inequality follows from $\theta = b/a_I$ for $d = 1$ and $\delta\theta < 1$. For $x' \in (x_{m-1}, x_m]$ for some $m \in \{1, 2, \dots, n\}$,

$$\begin{aligned}
\frac{\partial W_0(x, x')}{\partial x'} &= -\frac{a_I}{a_C b} + \delta\left(-\frac{\sum_{i=0}^{m-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^m}{a_C}\right) \\
&\leq -\frac{a_I}{a_C b} + \delta\left(-\frac{1}{a_I - a_C} - \frac{\zeta\delta}{a_C}\right) \\
&< -\frac{a_I}{a_C b} + \frac{\delta}{a_C} = \frac{a_I(\delta\theta - 1)}{a_C b} < 0,
\end{aligned}$$

where the first inequality follows from Lemma 3, the second inequality follows from $\delta\theta < 1$ and Lemma A4 with $d = 1$, the third inequality follows from $\delta\theta < 1$. For $x' > x_n$, we have $\partial W_0(x, x')/\partial x' = -a_I/(a_C b) < 0$. Thus, for any x', $W_0(x, x')$ strictly decreases with x', so $W_0(x, x')$ attains its maximum when x' attains its minimum: $x' = (1 - d)x = 0$ for $x \in (0, a_C]$. In what follows, similar to the proof of Proposition 3, we focus on $x' \geq \zeta(\hat{x} - x) + \hat{x}$ for $x > a_C$. Using the reduced-form utility function, we have

$$W_0(x, x') \equiv u(x, x') + \delta W(x') = -\frac{1}{b}x' + 1 + \delta W(x').$$

Consider $x \in (x_{m-1}, x_m]$ with $m \in \{1, 2, \dots, n\}$. Since $\zeta < 0$, $x' \leq \zeta(\hat{x} - x) + \hat{x} \leq \zeta(\hat{x} - x_m) + \hat{x} = x_{m-1}$, where the last equality follows from the construction of x_m. If $x' < a_C$, then,

$$\frac{\partial W_0(x, x')}{\partial x'} = -\frac{1}{b} + \frac{\delta}{a_C} > 0,$$

where the inequality follows from $b\delta > a_C$, or equivalently, $\delta > \mu_0 = 1/(b/a_C)$ with $d = 1$. If $x' \in (x_{m'-1}, x_{m'}]$ for some $m' \in \mathbb{N}$ and $m' \leq m - 1$, then

$$\frac{\partial W_0(x, x')}{\partial x'} = -\frac{1}{b} + \delta\left(-\frac{\sum_{i=0}^{m'-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^{m'}}{a_C}\right) = z_{m'}(\delta) > 0,$$

where the inequality follows from $m' \leq m - 1 \leq n - 1$, $\delta > \mu_{n-1} \geq \mu_{m'}$ (from Lemma 2), and $z_{m'}(\delta) > 0$ for $\delta > \mu_{m'}$ (from Lemma 2). Thus, for any $x' \leq \zeta(\hat{x} -$

$x) + \hat{x}$, $W_0(x, x')$ strictly increases with x' and for any $x' \geq \zeta(\hat{x} - x) + \hat{x}$, $W_0(x, x')$ strictly decreases with x', so $W_0(x, x')$ attains its maximum when $x' = \zeta(\hat{x} - x) + \hat{x}$.

Consider $x > x_n$. For $x' \in (0, a_C]$, following the argument for the previous case, we can show that $W_0(x, x')$ strictly increases with x'. For $x' \in (x_{m-1}, x_m]$ for some $m \in \{1, 2, \ldots, n\}$,

$$\frac{\partial W_0(x, x')}{\partial x'} = -\frac{1}{b} + \delta \left(-\frac{\sum_{i=0}^{m-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^m}{a_C} \right) = z_m(\delta). \tag{A.7}$$

Since $\mu_{n-1} < \delta < \mu_n$, from Lemma 2, $\mu_m \leq \mu_{n-1} < \delta$ for $m = 1, 2, \ldots, n-1$, and $\mu_m = \mu_n > \delta$ for $m = n$. From Lemma 2, $z_m(\delta) > 0$ for $m = 1, 2, \ldots, n-1$, and $z_m(\delta) < 0$ for $m = n$. Thus, $W_0(x, x')$ strictly increases with x' for $x_0 < x' \leq x_{n-1}$ and strictly decreases with x' for $x' \in (x_{n-1}, x_n]$. For $x' \in (x_n, \infty)$, $\partial W_0(x, x')/\partial x' = -1/b < 0$. Thus, we have shown that $W_0(x, x')$ strictly increases with x' for $x' < x_{n-1}$ and strictly decreases with x' for $x' > x_{n-1}$, so $W_0(x, x')$ attains its maximum when $x' = x_{n-1}$. Note that for $d = 1$ and $x > x_n$, (x, x_{n-1}) is always in Ω.

Last, we verify that the postulated value function is indeed consistent with the derived optimal policy function. For $x \in (0, a_C]$, $W(x) = x/a_C = u(x, 0) + \delta W(0)$. For $x \in (x_0, x_1] = (a_C, x_1]$,

$$\begin{aligned}
W(x) &= \left(-\frac{1}{a_I - a_C} - \frac{\delta\zeta}{a_C} \right)(x - \hat{x}) + \frac{(1-\delta)(a_I - \hat{x})}{(1-\delta)(a_I - a_C)} + \frac{\delta\hat{x}}{a_C} \\
&= \frac{a_I - x}{a_I - a_C} + \frac{\delta(\zeta(\hat{x} - x) + \hat{x})}{a_C} \\
&= u(x, \zeta(\hat{x} - x) + \hat{x}) + \delta W(\zeta(\hat{x} - x) + \hat{x}),
\end{aligned}$$

where the last equality follows from the reduced-form utility function and $(\zeta(\hat{x} - x) + \hat{x}) \in (0, a_C]$ for $x \in (x_0, x_1]$. For $x \in (x_{m-1}, x_m]$ with $m \in \{2, 3, \ldots, n\}$,

$$\begin{aligned}
W(x) &= \left(-\frac{\sum_{i=0}^{m-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^m}{a_C} \right)(x - \hat{x}) + \frac{(1-\delta^m)(a_I - \hat{x})}{(1-\delta)(a_I - a_C)} + \frac{\delta^m \hat{x}}{a_C} \\
&= \left(-\frac{1 + \sum_{i=1}^{m-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^m}{a_C} \right)(x - \hat{x}) + \frac{(1 - \delta + \delta - \delta^m)(a_I - \hat{x})}{(1-\delta)(a_I - a_C)} \\
&\quad + \frac{\delta^m \hat{x}}{a_C} = \frac{a_I - x}{a_I - a_C} + \left(-\frac{\sum_{i=1}^{m-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^m}{a_C} \right)(x - \hat{x}) \\
&\quad + \frac{(\delta - \delta^m)(a_I - \hat{x})}{(1-\delta)(a_I - a_C)} + \frac{\delta^m \hat{x}}{a_C} \\
&= \frac{a_I - x}{a_I - a_C} + \delta \left[\left(-\frac{\sum_{i=0}^{m-2}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^{m-1}}{a_C} \right)(\zeta(\hat{x} - x) + \hat{x} - \hat{x}) \right.
\end{aligned}$$

$$+\frac{(1-\delta^{m-1})(a_I-\hat{x})}{(1-\delta)(a_I-a_C)}+\frac{\delta^{m-1}\hat{x}}{a_C}\Bigg]$$
$$=u(x,\zeta(\hat{x}-x)+\hat{x})+\delta W(\zeta(\hat{x}-x)+\hat{x}),$$

where the last equality follows from the $(\zeta(\hat{x}-x)+\hat{x})\in(x_{m-2},x_{m-1}]$ for $x\in(x_{m-1},x_m]$ with $m\in\{2,3,\ldots,n\}$. For $x>x_n$ with $n=1$, we have

$$\begin{aligned}W(x)&=\frac{b-x_{n-1}}{b}+\frac{\sum_{i=1}^{1}\delta^i(a_I-x_{n-i})}{a_I-a_C}\\&=u(x,x_{n-1})+\delta=u(x,x_{n-1})+\delta W(x_{n-1}),\end{aligned}$$

where the last equality follows from $x_{n-1}=x_0=a_C$ for $n=1$ and $W(a_C)=1$. For $x>x_n$ with $n>1$, we have

$$\begin{aligned}W(x)&=\frac{b-x_{n-1}}{b}+\frac{\sum_{i=1}^{n}\delta^i(a_I-x_{n-i})}{a_I-a_C}\\&=u(x,x_{n-1})+\delta\left(\frac{\sum_{i=0}^{n-2}\delta^i(a_I-x_{n-i-1})}{a_I-a_C}+\delta^{n-1}\right)\\&=u(x,x_{n-1})+\delta\Bigg(\frac{\sum_{i=0}^{n-2}\delta^i[a_I-\hat{x}+(-\zeta)^i(\hat{x}-x_{n-1})]}{a_I-a_C}\\&\quad+\frac{\delta^{n-1}(a_C-\hat{x})}{a_C}+\frac{\delta^{n-1}\hat{x}}{a_C}\Bigg)\\&=u(x,x_{n-1})+\delta\Bigg(\left(-\frac{\sum_{i=0}^{n-2}(-\delta\zeta)^i}{a_I-a_C}+\frac{(-\delta\zeta)^{n-1}}{a_C}\right)(x_{n-1}-\hat{x})\\&\quad+\frac{(1-\delta^{n-1})(a_I-\hat{x})}{(1-\delta)(a_I-a_C)}+\frac{\delta^{n-1}\hat{x}}{a_C}\Bigg)\\&=u(x,x_{n-1})+\delta W(x_{n-1}).\end{aligned}$$

Thus, we have shown that $W(\cdot)$ satisfies the Bellman equation and the optimal policy is given by

$$g(x)=\begin{cases}0 & \text{for } x\in(0,a_C]\\ -\zeta x+\frac{a_C b}{a_C-a_I} & \text{for } x\in(a_C,x_n]\\ x_{n-1} & \text{for } x\in(x_n,\infty)\end{cases}.$$

□

Proof of Propositions 6 and 7 There are two possible cases: (i) $\mu_0\geq 1$ and (ii) $\mu_0<1$. For (i), since $\mu_0>1$, we always have $\delta<1\leq\mu_0$. Then Theorem 2 applies. For (ii), from Lemma A5, we know there exists a unique $n_0\in\mathbb{N}$ such that $\mu_{n_0-1}<1\leq\mu_{n_0}$ and $x_{n_0}<a_I$. Since $x_{n_0}<a_I$, the optimal policy functions stated in the propositions are properly defined for $\delta\in(\mu_{n_0-1},1)\subset(\mu_{n_0-1},\mu_{n_0})$. Then, following essentially

the same argument as in the proofs of Propositions 3 and 4, we can obtain the optimal policy. ◻

Proof of Proposition 8 For $0 < d < 1$, following the same argument as the proof of Theorem 1, we can show the value function $V(\cdot)$ is again given by (A.1) for $\delta = 1/\theta$. Since $\delta = 1/\theta$, the inequality in (A.2) for Case (i) in the proof of Theorem 1 becomes an equality. Then, we can establish that the optimal policy correspondence $h(x) = [(1-d)x, \min\{a_C, \theta x\}]$ for $x \in (0, a_I]$, $h(x) = [(1-d)x, \min\{a_C, -\zeta x + \frac{a_C}{a_C - a_I}\}]$ for $x \in (a_I, a_C]$, and $h(x) = \{(1-d)x\}$ for $x > a_C$. For $d = 1$, a similar argument can be applied to obtain the optimal policy correspondence. ◻

Proof of Proposition 9 Let $\delta = 1/\theta$ with $\theta > 1$ and $0 < d < 1$. The proof follows closely the proof of Theorem 1 in Fujio et al. (2021). Postulate a candidate value function given by

$$W(x) = \begin{cases} \frac{a_I \theta}{a_C b}\delta^n(\theta^n x - \hat{x}) + \frac{\delta^n}{1-\delta}u(\hat{x}, \hat{x}) & \text{for } x \in [\frac{\hat{x}}{\theta^{n+1}}, \frac{\hat{x}}{\theta^n}) \\ \frac{1-d}{b}\delta^n\left[(1-d)^n x - \hat{x}\right] + \frac{1-\delta^n + \delta^n u(\hat{x},\hat{x})}{1-\delta} & \text{for } x \in [\frac{\hat{x}}{(1-d)^n}, \frac{\hat{x}}{(1-d)^{n+1}}) \end{cases}$$

where $n = 0, 1, 2 \ldots$ We now verify if $W(x)$ satisfies the Bellman equation. We consider three cases: (i) $x \in (0, \hat{x})$; (ii) $x \in [\hat{x}, \frac{\hat{x}}{1-d})$; (iii) $x \in [\frac{\hat{x}}{(1-d)^n}, \frac{\hat{x}}{(1-d)^{n+1}})$ with $n \geq 1$.

For Case (i), there exists $n \in \mathbb{N}$ such that $x \in [\hat{x}/\theta^n, \hat{x}/\theta^{n-1})$. Pick x' such that $(x, x') \in \Omega$. If $x' > \hat{x} \geq \zeta(\hat{x} - x) + \hat{x}$, then there exists $n_0 \in \mathbb{N}$ such that $x' \in [\frac{\hat{x}}{(1-d)^{n_0-1}}, \frac{\hat{x}}{(1-d)^{n_0}})$. Since $x' \geq \zeta(\hat{x} - x) + \hat{x}$, we have

$$W_0(x, x') \equiv u(x, x') + \delta W(x') = \frac{a_I \theta}{a_C b}x - \frac{a_I}{a_C b}x' + \delta W(x');$$
$$\frac{\partial W_0(x, x')}{\partial x'} = \frac{1}{a_C b}\left[a_C \delta^{n_0}(1-d)^{n_0} - a_I\right] < 0,$$

where the inequality follows from $a_C < a_I$ and $\delta(1-d) < 1$. Consider x' such that $\hat{x} > x' \geq \zeta(\hat{x} - x) + \hat{x}$. There exists $n_0 \in \mathbb{N}$ such that $x' \in [\frac{\hat{x}}{\theta^{n_0}}, \frac{\hat{x}}{\theta^{n_0-1}})$. Since $x' \geq \zeta(\hat{x} - x) + \hat{x}$,

$$W_0(x, x') \equiv u(x, x') + \delta W(x') = \frac{a_I \theta}{a_C b}x - \frac{a_I}{a_C b}x' + \delta W(x');$$
$$\frac{\partial W_0(x, x')}{\partial x'} = \frac{a_I}{a_C b}\left[(\delta\theta)^{n_0} - 1\right] = 0,$$

where the last equality follows from $\delta = 1/\theta$. For $x' < \zeta(\hat{x} - x) + \hat{x} < \hat{x}$, there exists $n_0 \in \mathbb{N}$ such that $x' \in [\frac{\hat{x}}{\theta^{n_0}}, \frac{\hat{x}}{\theta^{n_0-1}})$. Since $x' < \zeta(\hat{x} - x) + \hat{x}$,

$$W_0(x, x') \equiv u(x, x') + \delta W(x') = \frac{1-d}{b}x - \frac{1}{b}x' + 1 + \delta W(x');$$
$$\frac{\partial W_0(x, x')}{\partial x'} = \frac{1}{b}\left[\frac{a_I}{a_C}(\delta\theta)^{n_0} - 1\right] > 0,$$

where the inequality follows from $\delta\theta = 1$ and $a_I > a_C$. Taken together, we have shown that $W_0(x, x')$ strictly decreases with x' for $x' > \hat{x}$, strictly increases with x' for $x' < \zeta(\hat{x} - x) + \hat{x}$, and is constant with respect to x' for $x' \in [\zeta(\hat{x} - x) + \hat{x}, \hat{x}]$. Since $(x, x') \in \Omega$, $\theta x \geq x' \geq (1 - d)x$. Thus, $W_0(x, x')$ is maximized for $x' \in [\max\{(1 - d)x, -\zeta x + \frac{a_C b}{a_C - a_I}\}, \min\{\theta x, \hat{x}\}]$.

For Cases (ii) and (iii), following the proof of Theorem 1 in Fujio et al. (2021), we can show that for $x \in [\hat{x}, \frac{\hat{x}}{1-d})$, $W_0(x, x')$ is maximized with $x' = \hat{x}$, and for $x \in [\frac{\hat{x}}{(1-d)^n}, \frac{\hat{x}}{(1-d)^{n+1}})$ with $n \in \mathbb{N}$, $W_0(x, x')$ is maximized with $x' = (1 - d)x$. Consider the policy correspondence

$$\bar{h}(x) = \begin{cases} [\max\{(1-d)x, -\zeta x + \frac{a_C b}{a_C - a_I}\}, \theta x] & \text{for } x \in (0, \frac{\hat{x}}{\theta}] \\ [\max\{(1-d)x, -\zeta x + \frac{a_C b}{a_C - a_I}\}, \hat{x}] & \text{for } x \in (\frac{\hat{x}}{\theta}, \hat{x}] \\ \{\hat{x}\} & \text{for } x \in (\hat{x}, \frac{\hat{x}}{1-d}] \\ \{(1-d)x\} & \text{for } x \in (\frac{\hat{x}}{1-d}, \infty) \end{cases},$$

and the straight-down-the-turnpike policy

$$\bar{g}(x) = \begin{cases} \theta x & \text{for } x \in (0, \frac{\hat{x}}{\theta}] \\ \hat{x} & \text{for } x \in (\frac{\hat{x}}{\theta}, \frac{\hat{x}}{1-d}] \\ (1-d)x & \text{for } x \in (\frac{\hat{x}}{1-d}, \infty) \end{cases}.$$

For any $x > 0$, $\bar{g}(x) \in \bar{h}(x)$, and as shown above, $W_0(x, x')$ is maximized for any $x' \in \bar{h}(x)$ and in particular, for $x' = \bar{g}(x)$. Since $W(x) = u(x, \bar{g}(x)) + \delta W(\bar{g}(x))$ for any x (as in Fujio et al. (2021)), $W(x) = u(x, x') + \delta W(x')$ for any $x' \in \bar{h}(x)$. Thus, $W(\cdot)$ satisfies the Bellman equation and the optimal policy correspondence is given by $\bar{h}(\cdot)$. □

Proof of Proposition 10 For $\delta < 1/\theta$, the proof follows the proof of Theorem 1, so the optimal policy for $a_C = a_I$ is also given by $g(x) = (1 - d)x$ for any $x > 0$. For $\delta = 1/\theta$, the proof of Proposition 8 also carries over to the one-sector case ($a_C = a_I$) but with one modification: for $a_C = a_I$ and $\delta = 1/\theta$, both inequalities in (A.3) become equalities. This implies for $x > a_C$, the optimal policy is given by $h(x) = [(1 - d)x, \max\{a_C, (1 - d)x\}]$. □

Proof of Propositions A1 and A2 We first consider the case of $0 < d < 1$. For $\delta = \mu_0$, we can follow the proof of Theorem 2 to establish the optimal policy correspondence. The only difference is that the inequality (A.4) becomes equality for $\delta = \mu_0$. This implies that for $x \in (a_C, \frac{a_C}{1-d}]$, $\mathcal{W}_0(x, x')$ is maximized for any x' such that $x' \geq (1 - d)x$, $x' \leq \zeta(\hat{x} - x) + \hat{x}$, and $x' \leq a_C$. We thus establish Proposition A1(i). For $\delta = \mu_n$ for some $n \in \mathbb{N}$, we can follow the proof Proposition 3. The only difference is that for (A.6), $\frac{\partial W_0(x,x')}{\partial x'} = 0$ for $m = n$ because $\delta = \mu_n$. Thus, for $x \in (x_n, \frac{x_{n-1}}{1-d}]$, $W_0(x, x')$ strictly increases with x' for $x' \leq x_{n-1}$, is constant with respect to x' for $x' \in (x_{n-1}, x_n]$, and strictly decreases with x' for $x' > x_n$. Then, $W_0(x, x')$ is maximized for $x' \in (x_{n-1}, x_n]$. Similarly, we can show that for $x \in (\frac{x_{n-1}}{1-d}, \frac{x_n}{1-d}]$, $W_0(x, x')$ is maximized for $x' \in (x_{n-1}, x_n]$. Using the fact that $(x, x') \in \Omega$, we then obtain the optimal policy correspondence as in Proposition A1(ii). The argument is essentially the same for the case of $d = 1$. □

A.3 Auxiliary results

Lemma A1 *For any non-negative integer n, we can write $z_n(\delta)$ as*

$$z_n(\delta) = \begin{cases} \frac{ba_I(-\zeta)^n(1-\theta\delta)\delta^{n+1} - a_C(a_I-a_C)(1-(1-d)\delta)^2}{a_C b(1-\delta(1-d))(a_I-a_C)(1+\delta\zeta)} & \text{for } \delta \neq -\frac{1}{\zeta} \\ \frac{-nba_C(1-d)\delta + nba_C - ba_I + 2ba_C}{a_C b(1-\delta(1-d))(a_I-a_C)\zeta} & \text{for } \delta = -\frac{1}{\zeta} \end{cases}. \tag{A.8}$$

Proof For $\delta \neq -1/\zeta$, $-\delta\zeta \neq 1$ and we can simplify the geometric series in $z_n(\delta)$ as follows

$$\begin{aligned} z_n(\delta) &= -\frac{1}{b} + \delta\left(-\frac{\sum_{i=0}^{n-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^n}{a_C(1-\delta(1-d))}\right) \\ &= -\frac{1}{b} + \delta\left(-\frac{1-(-\delta\zeta)^n}{(1+\delta\zeta)(a_I-a_C)} + \frac{(-\delta\zeta)^n}{a_C(1-\delta(1-d))}\right) \\ &= \frac{-(1+\delta\zeta)(a_I-a_C)a_C(1-\delta(1-d)) - ba_C(1-\delta(1-d))\delta(1-(-\delta\zeta)^n))}{a_C b(1-\delta(1-d))(a_I-a_C)(1+\delta\zeta)} \\ &\quad + \frac{\delta b(1+\delta\zeta)(a_I-a_C)(-\delta\zeta)^n}{a_C b(1-\delta(1-d))(a_I-a_C)(1+\delta\zeta)} \\ &= \frac{ba_I(-\zeta)^n(1-\theta\delta)\delta^{n+1} - a_C(a_I-a_C)(1-(1-d)\delta)^2}{a_C b(1-\delta(1-d))(a_I-a_C)(1+\delta\zeta)}, \end{aligned}$$

where we note that even though the first equation does not apply to $n = 0$ because the summation is not property defined, the expressions in the second line onward apply for any non-negative integer n.

For $\delta = -1/\zeta$, $-\delta\zeta = 1$, we have

$$\begin{aligned} z_n(\delta) &= -\frac{1}{b} + \delta\left(-\frac{\sum_{i=0}^{n-1}(-\delta\zeta)^i}{a_I - a_C} + \frac{(-\delta\zeta)^n}{a_C(1-\delta(1-d))}\right) \\ &= -\frac{1}{b} + \delta\left(-\frac{n}{a_I - a_C} + \frac{1}{a_C(1-\delta(1-d))}\right) \\ &= \frac{-(a_I-a_C)a_C(1-\delta(1-d)) - bna_C(1-\delta(1-d))\delta + b(a_I-a_C)\delta}{a_C b(1-\delta(1-d))(a_I-a_C)} \\ &= \frac{-nba_C(1-d)\delta + nba_C - ba_I + 2ba_C}{a_C b(1-\delta(1-d))(a_I-a_C)\zeta}, \end{aligned}$$

where again we note that even though the first equation does not apply to $n = 0$, the expressions in the second line onward apply for any non-negative integer n. We have thus obtained the desired conclusion. □

Lemma A2 *For any positive integer n, define* $\tilde{z}_n : [0, 1/\theta] \to \mathbb{R}$ *given by*

$$\tilde{z}_n(\delta) \equiv ba_I(-\zeta)^n(1-\theta\delta)\delta^{n+1} - a_C(a_I - a_C)(1-(1-d)\delta)^2. \tag{A.9}$$

For $\delta \neq -1/\zeta$, $z_n(\delta) = 0$ *if and only if* $\tilde{z}_n(\delta) = 0$. *Moreover, if* $a_I > a_C$, *then* $\tilde{z}_n(\cdot)$ *and its derivatives satisfy:*

(a.1) $\tilde{z}_n(-1/\zeta) = 0$, *(a.2)* $\tilde{z}_n(0) < 0$, *(a.3)* $\tilde{z}_n(1/\theta) < 0$.

(b) $z_n\,(-1/\zeta)\,\tilde{z}_n'(-1/\zeta) \le 0$ *and* $\tilde{z}_n'(-1/\zeta) = 0$ *if and only if* $z_n(-1/\zeta) = 0$.

(c) There exists $\bar{\delta} \in (0, 1/\theta)$ *such that* $\tilde{z}_n'(\delta) > 0$ *for* $\delta \in [0, \bar{\delta})$ *and* $\tilde{z}_n'(\delta) < 0$ *for* $\delta \in (\bar{\delta}, 1/\theta]$.

Proof From Lemma A1, for $\delta \neq -1/\zeta$,

$$z_n(\delta) = \frac{ba_I(-\zeta)^n(1-\theta\delta)\delta^{n+1} - a_C(a_I - a_C)(1-(1-d)\delta)^2}{a_C b(1-\delta(1-d))(a_I - a_C)(1+\delta\zeta)},$$

so $z_n(\delta) = 0$ if the only if

$$ba_I(-\zeta)^n(1-\theta\delta)\delta^{n+1} - a_C(a_I - a_C)(1-(1-d)\delta)^2 = \tilde{z}_n(\delta) = 0.$$

Let $a_I > a_C$. Since $\tilde{z}_n(\cdot)$ is continuous on $[0, 1/\theta]$, we have

$$\begin{aligned}
\tilde{z}_n(-1/\zeta) &= \lim_{\delta \to -1/\zeta} \tilde{z}_n(\delta) \\
&= \lim_{\delta \to -1/\zeta} a_C b(1-\delta(1-d))(a_I - a_C)(1+\delta\zeta)z_n(\delta) \\
&= \lim_{\delta \to -1/\zeta} a_C b(1-\delta(1-d))(a_I - a_C)(1+\delta\zeta)z_n(-1/\zeta) = 0,
\end{aligned}$$

where the second equality follows from the definition of $\tilde{z}_n(\cdot)$ and Lemma A1, the third equality follows from the continuity of $z_n(\cdot)$ on $[0, 1/\theta]$, and the last equality follows from $(1+\delta\zeta) = 0$ for $\delta = -1/\zeta$. Since $a_I > a_C > 0$, $\tilde{z}_n(0) = -a_C(a_I - a_C) < 0$. Since $\theta > (1-d)$ and $a_I > a_C > 0$, $\tilde{z}_n(1/\theta) = -a_C(a_I - a_C)(1-(1-d)/\theta)^2 < 0$. Thus, we have established (a.1)–(a.3).

For (b), since

$$\begin{aligned}
\tilde{z}_n'(\delta) = {} & ba_I(-\zeta)^n\left[(n+1)(1-\theta\delta)\delta^n - \theta\delta^{n+1}\right] \\
& + 2a_C(a_I - a_C)(1-d)(1-(1-d)\delta),
\end{aligned} \tag{A.10}$$

we have

$$\begin{aligned}
\tilde{z}_n'\left(-\frac{1}{\zeta}\right) &= (n+1)ba_I\left(1+\frac{\theta}{\zeta}\right) + \frac{ba_I\theta}{\zeta} + 2a_C(a_I - a_C)\left(1+\frac{1-d}{\zeta}\right)(1-d) \\
&= \frac{1}{\zeta}\left[\frac{(n+1)b^2a_C}{a_C - a_I} + b^2 + b(1-d)a_I - 2a_C b(1-d)\right] \\
&= \frac{1}{\zeta}\left[nba_C(\zeta + (1-d)) + \frac{b^2a_C}{a_C - a_I} + b^2 + b(1-d)(a_I - 2a_C)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\zeta}\left[nba_C\zeta + (1-d)nba_C + \left(-\frac{b^2}{a_C - a_I} + b(1-d)\right)(a_I - 2a_C)\right]\\
&= \frac{n(1-d)ba_C}{\zeta} + nba_C - b(a_I - 2a_C),\\
&= a_C b(1 + (1-d)/\zeta)(a_I - a_C)\zeta z_n\left(-\frac{1}{\zeta}\right),
\end{aligned}
$$

where the last equality follows from Lemma A1 for $\delta = -1/\zeta$. Since $\tilde{z}_n'(-1/\zeta) = a_C b(1+(1-d)/\zeta)(a_I - a_C)\zeta z_n(-1/\zeta)$, $\tilde{z}_n'(-1/\zeta) = 0$ if and only if $z_n(-1/\zeta) = 0$. Moreover, since $\zeta < 0$ (for $a_I > a_C$) and $a_I > a_C$,

$$
z_n\left(-\frac{1}{\zeta}\right)\tilde{z}_n'\left(-\frac{1}{\zeta}\right) = a_C b(1-\delta(1-d))(a_I - a_C)\zeta z_n^2\left(-\frac{1}{\zeta}\right) \le 0.
$$

Thus, we have established (b).

For (c), define $f(\delta) \equiv ba_I(-\zeta)^n\left[(n+1)(1-\theta\delta)\delta^n - \theta\delta^{n+1}\right]$. Then, from Eq. (A.10), we have

$$
\tilde{z}_n'(\delta) = f(\delta) + 2a_C(a_I - a_C)(1-d)(1-(1-d)\delta) > f(\delta),
$$

where the inequality follows from $a_I > a_C$ and $\delta \in [0, 1/\theta]$. Since $f'(\delta) = ba_I(-\zeta)^n(n+1)\delta^{n-1}[n-(n+2)\theta\delta]$, $f'(\delta) \ge 0$ for $\delta \in [0, \frac{n}{(n+2)\theta}]$ and $f'(\delta) < 0$ for $\delta > \frac{n}{(n+2)\theta}$. Since $f(0) = 0$ and $f'(\delta) \ge 0$ for $\delta \in [0, \frac{n}{(n+2)\theta}]$, $f(\delta) \ge 0$ for $\delta \in [0, \frac{n}{(n+2)\theta}]$. Then, $\tilde{z}_n'(\delta) > f(\delta) \ge 0$ for $\delta \in [0, \frac{n}{(n+2)\theta}]$. For $\delta > \frac{n}{(n+2)\theta}$, $\tilde{z}_n''(\delta) = f'(\delta) - 2a_C(a_I - a_C)(1-d)^2 < f'(\delta) < 0$, which implies that $\tilde{z}_n'(\cdot)$ is strictly decreasing on the interval $[\frac{n}{(n+2)\theta}, \frac{1}{\theta}]$. Suppose $\tilde{z}_n'(1/\theta) \ge 0$. By the monotonicity, we must have $\tilde{z}_n'(\delta) > 0$ for $\delta \in (\frac{n}{(n+2)\theta}, \frac{1}{\theta})$ and we have shown that $\tilde{z}_n'(\delta) > 0$ for $\delta \in [0, \frac{n}{(n+2)\theta}]$, so $\tilde{z}_n(\cdot)$ is strictly increasing on $\delta \in [0, 1/\theta]$. However, from (a.1) and (a.3), we know $\tilde{z}_n(-1/\zeta) = 0 > \tilde{z}_n(1/\theta)$ with $1/\theta > -1/\zeta$, contradicting to $\tilde{z}_n(\cdot)$ being strictly increasing. Thus, we must have $\tilde{z}_n'(1/\theta) < 0$. Since $\tilde{z}_n'(\cdot)$ is strictly decreasing on the interval $[\frac{n}{(n+2)\theta}, \frac{1}{\theta}]$ and $\tilde{z}_n'(\frac{n}{(n+2)\theta}) > 0$, by the continuity of $\tilde{z}_n'$, there exists $\bar{\delta} \in (\frac{n}{(n+2)\theta}, \frac{1}{\theta})$ such that $\tilde{z}_n'(\bar{\delta}) = 0$, $\tilde{z}_n'(\delta) < 0$ for $\delta > \bar{\delta}$ and $\tilde{z}_n'(\delta) > 0$ for $\delta \in [\frac{n}{(n+2)\theta}, \bar{\delta})$. Since we have already shown that $\tilde{z}_n'(\delta) > 0$ for $\delta \in [0, \frac{n}{(n+2)\theta}]$, we have obtained the desired conclusion. □

Lemma A3 *For $\zeta \ne -1$, we can express x_n more explicitly as*

$$
x_n = \frac{a_C b}{b + d(a_C - a_I)} - \frac{da_C(a_I - a_C)}{(b + d(a_C - a_I))(-\zeta)^n}.
$$

For $\theta > 1$, we further have $x_n = \hat{x} - (\hat{x} - a_C)/(-\zeta)^n$. Moreover, for $\zeta = -1$, $x_n = a_C + na_C b/(a_I - a_C)$.

Proof From Eq. (A.5) in the proof of Lemma 4, we directly obtain

$$x_n = \frac{a_C b}{b + d(a_C - a_I)} - \frac{d a_C (a_I - a_C)}{(b + d(a_C - a_I))(-\zeta)^n}.$$

For $\theta > 1$, we have $\hat{x} = \frac{a_C b}{b+d(a_C - a_I)}$, so we can further simplify the expression above to obtain

$$x_n = \hat{x} - (\hat{x} - a_C)/(-\zeta)^n.$$

For $\zeta = -1$, $x_n = x_{n-1} + a_C b/(a_I - a_C) = x_0 + n a_C b/(a_I - a_C) = a_C + n a_C b/(a_I - a_C)$, where the last equality follows from $x_0 = a_C$. □

Lemma A4 *If $\delta\theta < 1$ and $a_C < a_I$, then $-\frac{1}{a_I - a_C} - \frac{\zeta\delta}{a_C(1-\delta(1-d))} < \frac{1}{a_C(1-\delta(1-d))}$.*

Proof Since $\delta\theta < 1$ and $\theta = b/a_I + (1 - d)$, we have

$$\begin{aligned}
\delta\theta < 1 &\Leftrightarrow \delta(b + a_I(1-d)) - a_I < 0 \\
&\Leftrightarrow \delta(b - (a_C - a_I)(1-d)) + \delta a_C(1-d) + (a_C - a_I) - a_C < 0 \\
&\Leftrightarrow (a_C - a_I)\zeta\delta + (a_C - a_I) - a_C(1 - \delta(1-d)) < 0 \\
&\Leftrightarrow (-\zeta\delta - 1)(a_I - a_C) - a_C(1 - \delta(1-d)) < 0 \\
&\Leftrightarrow \frac{-\zeta\delta - 1}{a_C(1-\delta(1-d))} < \frac{1}{a_I - a_C} \\
&\Leftrightarrow -\frac{1}{a_I - a_C} - \frac{\zeta\delta}{a_C(1-\delta(1-d))} < \frac{1}{a_C(1-\delta(1-d))},
\end{aligned}$$

where the second to last line follows from $a_C < a_I$ and $\delta(1-d) < 1$. □

Lemma A5 *Let $a_C < a_I$, $\theta < 1$, and $\mu_0 < 1$. There exists a unique $n_0 \in \mathbb{N}$ such that $\mu_{n_0 - 1} < 1 \leq \mu_{n_0}$ and $x_{n_0} < a_I$.*

Proof Since $\theta < 1$, from Lemma 2, $\lim_{n\to\infty} \mu_n = 1/\theta > 1$. We claim that there exists a unique $n_0 \in \mathbb{N}$ such that $\mu_{n_0-1} < 1 \leq \mu_{n_0}$. Suppose on the contrary, there does not exist a natural number n_0 such that $\mu_{n_0-1} < 1 \leq \mu_{n_0}$. Since the sequence $\{\mu_n\}_{n=0}^{\infty}$ is monotonically increasing and $\mu_0 < 1$, this implies that $\mu_n < 1$ for any $n \in \mathbb{N}$. Since $\mu_n < 1$ for any n, $\lim_{n\to\infty} \mu_n \leq 1$, leading to a contradiction. The strict monotonicity of $\{\mu_n\}_{n=0}^{\infty}$ further guarantees the uniqueness of n_0. What remains to show is that $x_{n_0} < a_I$.

Since $\mu_{n_0-1} < 1 \leq \mu_{n_0}$, from Lemma 2, we have $z_{n_0-1}(1) > 0$. For $\zeta = -1$, from Lemma A1, we have

$$z_{n_0-1}(1) = \frac{(n_0 - 1) b a_C d - b a_I + 2 b a_C}{a_C b d (a_I - a_C)\zeta} > 0 \;\Leftrightarrow\; (n_0 - 1) a_C d < a_I - 2a_C,$$

where the second inequality follows from $a_I > a_C$ and $\zeta < 0$. From Lemma A3, for $\zeta = -1$,

$$\begin{aligned} x_{n_0} &= a_C + \frac{n_0 a_C b}{a_I - a_C} \\ &= a_C + n_0 a_C d < a_C + a_C d + a_I - 2a_C = a_I - (1-d)a_C < a_I, \end{aligned}$$

where the second equation follows from $\zeta = -1$ and the first inequality follows from $(n_0 - 1)a_C d < a_I - 2a_C$. For $\zeta \neq -1$, from Lemma 2, we have

$$\begin{aligned} z_{n_0-1}(1) &= \frac{b a_I (-\zeta)^{n_0-1}(1-\theta) - a_C(a_I - a_C)d^2}{a_C b d (a_I - a_C)(1+\zeta)} > 0 \\ &\Leftrightarrow \frac{a_C(a_I - a_C)d}{(-\zeta)^{n_0}(1+\zeta)} < \frac{b a_I (1-\theta)}{-\zeta d(1+\zeta)} \\ &\Leftrightarrow \frac{d a_C (a_I - a_C)}{(b + d(a_C - a_I))(-\zeta)^{n_0}} > \frac{b a_I (1-\theta)}{-\zeta d(b + d(a_C - a_I))}, \end{aligned} \tag{A.11}$$

where the last inequality follows from $a_C < a_I$. Further, from Lemma A3, we have

$$\begin{aligned} x_{n_0} &= \frac{a_C b}{b + d(a_C - a_I)} - \frac{d a_C (a_I - a_C)}{(b + d(a_C - a_I))(-\zeta)^{n_0}} \\ &< \frac{a_C b}{b + d(a_C - a_I)} - \frac{b a_I (1-\theta)}{-\zeta d(b + d(a_C - a_I))} \\ &= \frac{a_C b}{b + d(a_C - a_I)} - \frac{b(d a_I - b)}{-\zeta d(b + d(a_C - a_I))} \\ &= \frac{a_C b}{b + d(a_C - a_I)} - \frac{b(d a_I - b)}{d(b + d(a_C - a_I))} + \frac{b(d a_I - b)}{d(b + d(a_C - a_I))} \\ &\quad - \frac{b(d a_I - b)}{-\zeta d(b + d(a_C - a_I))} \\ &= \frac{a_C b d - a_I b d + b^2}{(b + d(a_C - a_I))d} + \frac{(\zeta + 1) b (d a_I - b)}{\zeta d(b + d(a_C - a_I))} \\ &= \frac{b}{d} + \frac{b(d a_I - b)}{\zeta d(a_C - a_I)} = a_I + \frac{b - a_I d}{d} + \frac{b(d a_I - b)}{\zeta d(a_C - a_I)} \\ &= a_I + \frac{(b - a_I d)(\zeta(a_C - a_I) - b)}{\zeta d(a_C - a_I)} = a_I - \frac{(b - a_I d)(1-d)}{\zeta d} \le a_I, \end{aligned}$$

where the first inequality follows from (A.11) and the last inequality follows from $\zeta < 0$ and $b < a_I d$ (from $\theta < 1$). Then, we have obtained the desired conclusion. □

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