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The miniature two-sector model of optimal growth: The neglected case of a capital-intensive investment-good sector☆

Minako Fujio[a,*], Yan Lei[b], Liuchun Deng[c], M. Ali Khan[d]

[a] Department of Economics, Yokohama National University, Japan
[b] Department of Economics, National University of Singapore, Singapore
[c] Division of Social Sciences, Yale-NUS College, Singapore
[d] Department of Economics, The Johns Hopkins University, Baltimore, MD 21218, United States

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ABSTRACT

This paper considers the neglected case of a capital-intensive investment-good sector in the two-sector Robinson-Shinkai-Leontief (RSL) model of discrete-time optimal economic growth. We find the optimal policy to be surprisingly simple and uniform between the discounted and undiscounted cases. The "straight-down-the-turnpike" policy, first identified by Winter and Shell, entails unemployment or excess supply of capital throughout the optimal transition dynamics. We extend our analysis to the case of circulating capital and find the optimal policy to have the same property. We also briefly indicate possibilities for application to topical concerns.

> For almost two decades, from the early 1950s through the late 1960s, growth theory dominated economic theory, and Bob Solow dominated growth theory. Seldom, if ever, has a single paradigm, a particular set of questions and a particular approach to answering those questions, held such sway over a discipline[1] Stiglitz (1990).

> The more I studied Mahalanobis's work, the more I felt somewhat uneasy about this relationship between the stability of the dynamic system and factor-price intensities of consumption goods and investment goods. This was really important, not only in market economies, but also in planned economies[2] Uzawa (1998).

☆ This paper is dedicated to Wahidul Haque as a token appreciation of his commitment to the use of mathematical economics in the cause of the economics of development. Without implicating them in any way, Khan thanks Ralph Chami, Duncan Foley, Geoff Harcourt, David Levy, Barkley Rosser, Eric Schliesser, Anwar Shaikh, Karl Shell and Kumaraswamy Velupillai for stimulating conversation and correspondence over the years. The authors gratefully acknowledge Haru Takahashi's kind sharing of his 2020 "Note on a continuous-time two-sector Leontief optimal growth model" with the authors and the encouragement of the Editor Daniela Puzzello and the careful reading of her two anonymous referees. Liuchun Deng acknowledges the support of the Start-up Grant from Yale-NUS College. The authors are listed in the *certified random order* of Ray-Robson (2018, American Economic Review).

* Corresponding author.
E-mail address: fujio-minako-rm@ynu.ac.jp (M. Fujio).

[1] Stiglitz (1999) continues, "I was lucky to have gone to MIT in the middle of this era. How can one describe the sense of excitement, the sense of making progress on a set of problems that were both intellectually challenging and, at least potentially, of considerable practical importance."

[2] This is from a June 29, 1998 interview by Okuno-Fujiwara and Shell (2009). Uzawa continues, "In fact, factor-price equalization was the first [time] that I became interested in capital intensities. At about that time, I used to know [Prasanta C.] Mahalanobis very well... and I was reading some of his earlier

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1. Introduction

In the charting of complicated dynamics arising from simple economic models, a two-sector model of discrete-time optimal economic growth, known as the Robinson-Shinkai-Leontief (RSL) model,[3] has been widely used as a workhorse. The model consists of a consumption-good and an investment-good sector, and its defining features are linear production technologies, a linear felicity function, inelastic labor supply, and irreversible investment. The existing literature focuses predominantly on the case of a capital-intensive consumption-good sector (Nishimura and Yano, 1995; Khan and Mitra, 2005; 2007),[4] and the case of a capital-intensive investment-good sector, arguably a more natural setting to begin with, is left largely under-explored.

In this paper, we provide a complete characterization of the optimal policy function for *this* neglected case of the two-sector RSL model with discounting. In particular, we demonstrate that the optimal policy is a straight-down-the-turnpike policy: the economy adjusts its capital stock as quickly as possible to the modified golden rule stock, even if full employment of labor has to be foregone or capacity of capital has to be under-utilized.[5] This result echoes the findings in Fujio (2005) concerning the same case of the RSL model but without discounting.[6] It is in stark contrast to the case of a capital-intensive consumption-good sector where such a straight-down-the-turnpike policy is optimal only when the discount factor is sufficiently high and additional parametric restrictions are imposed. Moreover, we further establish that this straight-down-the-turnpike policy remains optimal for two important special cases: (a) capital is circulating: this is to say that capital lasts for only one period; (b) capital intensity is the same for two sectors, which is the linear version of the so-called Ramsey-Cass-Koopmans (RCK) one-sector optimal growth model.

Despite the surprisingly sharp results our analysis engenders, perhaps because of them, we go beyond the narrow context of the two-sector growth model with linear technologies, and project the results presented here to two alternative frames. First, our results shed light on a long-standing discussion on how specialization and non-specialization are phased along an optimal path of two-sector models, a problem originally formulated in 1964 by Srinivasan (1964). Haque (1970) and Drabicki and Takayama (1975) provide a complete solution to the problem in a neoclassical setting where the vexatiousness of the case of capital-intensive investment-good sector arises from smooth production technology and therefore the tendency to full resource utilization. Second, the point of origin of the investigation of the RSL model is a continuous-time formulation of a development-planning choice of technique problem in Stiglitz (1968). Technically considerably more challenging as it is, our discrete-time reformulation abstracts from the problem of the choice of technique and thus shuts down an important theoretical pathway to economic discontinuity and complexity.[7]

As already mentioned, for the undiscounted case, Fujio (2005) showed that the optimal program is always monotonic and has, as one may conjecture with a linear utility function, "the most rapid approach property" (as formalized by Spence-Starrett) converging to the golden rule stock in a finite number of periods and almost always implies labor unemployment or excess capacity of capital along the transition path.[8] It is not at all apparent that this kind of simple optimal dynamics would continue to hold with discounting and that the kind of complex dynamics seen in the case where the consumption-good sector is more capital intensive would not reappear with some discounting. Multisector optimal growth models have in many cases generated complicated dynamics with moderate discounting. This paper provides a definite answer to this question.

In a continuous-time optimal growth model with the linear technology, Bruno (1967) fully characterizes the optimal dynamics for the two-sector case, a continuous-time analogue of the two-sector RSL model. Similar to our results in the discrete-time setting, when the investment-good sector is more capital intensive, the system converges as quickly as possible to the steady state with full employment only being achieved in the steady state. On the other hand, when the consumption-good sector is more capital intensive, the system monotonically converges to the steady state, contrasting with the emergence of cycles and chaos in the Robinson-Solow-Srinivasan (RSS) model, a polar case of the RSL model. Evidence for the dissonance between discrete- and continuous-time modelling is now clear from Khan and Mitra (2005) and Stiglitz (1968), on the one hand, and between Fujio (2006) and Uzawa (1964) and Haque (1970) on the other. What is of interest is that the

works. He had a four-sector model where the specifications of capital intensities among four sectors were very strange. So what I did was to formulate everything in terms of two-sector models. And this became rather serious over the real factor intensity problems in India; a very serious dispute was developed between [Sukhamoy] Chakravarty, the renowned economist, and [Bagicha Singh] Minhas." We also draw the reader's attention to the invaluable references Uzawa (1989, 1990) awaiting a serious exegesis.

[3] For the re-naming of what was previously termed the Leontief model in Fujio (2005, 2008, 2009) and its justifications, see Deng, Fujio, and Khan (2019).

[4] For more recent work, see Deng et al. (2019) and Deng and Fujio (2020). For the RSL model of equilibrium growth, see Deng, Fujio, and Khan (2020).

[5] To the authors' knowledge, this phrase, and the concept it represents, is due to Winter (1967) in the context of the von Neumann model and Shell (1967b) in the context of the Ramsey-Cass-Koopmans model with a linear felicity function. In the context of the Robinson-Solow-Srinivasan model, it was first used in Khan and Mitra (2006a). Also see Spence and Starrett (1975) for the "most rapid approach paths" in a class of continuous- and discrete-time dynamic optimization problems.

[6] In fact, Fujio (2005) is the only exception in the literature that tackles the RSL model with a capital-intensive investment-good sector.

[7] Rosser (1983), and the subsequent elaboration in Rosser (2000), demonstrates how a model with infinite and differentiable techniques still gives rise to discontinuity generated by reswitching.

[8] The relevance of the Spence and Starrett (1975) was already sighted in Fujio (2005, paragraph 3).

juxtaposition of the findings of this paper and the earlier results,[9] one obtains a more nuanced view of the sharpest rebuke, as in Khan and Mitra (2005), to the folk-wisdom that there is no substantive difference between discrete and continuous time: In the two-sector RSL model the point of divergence between discrete and continuous time dynamics actually *hinges on the assumption of capital intensity.*[10]

We have framed the results so far in the register of "pure theory," and more specifically in the case of linear technologies and therefore possibly in the narrower confines of non-neoclassical setting of optimal growth theory.[11] However, we would not want the reader to disregard our determined intention to take the model, and the analysis presented here, to the linear non-smooth continuous-time versions of the applied models in mathematical biosciences and evolutionary dynamics, and the associated economics literature.[12] Without intending to overstate the matter in any way, this may suggest a paradigmatic change in equilibrium and optimal economic dynamics, a change that goes back to May (1976). For the importance and the need for re-thinking of the subject, see Spence (1973) and Chami et al. (2020). May's 1976 article in *Nature* was already presented in the tone and the promise of a paradigmatic change.[13]

We present the standard set-up of the two-sector RSL model of optimal growth in Section 2, and the results on the optimal policy function in (a tripartite) Section 3. Since these results are derived through the use of methods of dynamic programming, they entail a guess-and-verify procedure. Our "guesses" and conjectures are based on numerical analysis and then elaborated diagrammatically. We summarize the results and state what to us are important open questions in Section 4. The detailed proofs are collected in the Appendix.[14]

2. The model

We consider the two-sector RSL model of optimal growth with discounting. There are two sectors: a consumption-good and an investment-good sector. The production technology is Leontief. One unit of consumption good requires one unit of labor and a_C units of capital, while b units of investment good require one unit of labor and a_I units of capital. We focus on the case of capital-intensive investment-good sector:

$$a_C < a_I. \tag{1}$$

Notably, this case is precluded in the RSS model with $a_I = 0$ as in Khan and Mitra (2005, 2007, 2012), and the RSL model with a capital-intensive consumption-good sector as in Nishimura and Yano (1995).

Labor supply is fixed and normalized to be one in each time period t. Denote the capital stock in the current period by x, the capital stock in the next period by x', and the depreciation rate of capital by $d \in (0, 1]$. The *transition possibility set* is given by

$$\Omega = \{(x, x') \in \mathbf{R}_+ \times \mathbf{R}_+ : x' - (1-d)x \geq 0, x' - (1-d)x \leq b \min\{1, x/a_I\}\},$$

where $\mathbf{R}_+$ is the set of non-negative real numbers. Denote by y the output of consumption good. For any $(x, x') \in \Omega$, we define a correspondence

$$\Lambda(x, x') = \{y \in \mathbf{R}_+ : 0 \leq y \leq (1/a_C)(x - (a_I/b)(x' - (1-d)x)) \text{ and } 0 \leq y \leq 1 - (1/b)(x' - (1-d)x)\}.$$

A felicity function, $w : \mathbf{R}_+ \longrightarrow \mathbf{R}$, is linear[15] and given by $w(y) = y$. The reduced form utility function, $u : \Omega \longrightarrow \mathbf{R}_+$, is defined as

$$u(x, x') = \max\{w(y) : y \in \Lambda(x, x')\}.$$

The future utility is discounted with a discount factor $\rho \in (0, 1)$.

[9] Including those of Bruno (1967) and Khan and Mitra (2012, 2020) and their references.

[10] We are indebted to the referee for this observation: the italicization follows his/her recommendation.

[11] The two epigraphs to this work testify to this; also see Meade (1961). His "neoclassical theory of economic growth" is an important neoclassical counterpoint to our work. Whereas we are clear that there is much yet to learn from Meade's analysis of the smooth marginal case, the point is that in the model presented here the elasticity of substitution between capital and labour in either the consumption or the investment-good sector is zero, and plays no subsequent role in the analysis; see Meade's Appendices I and II, and also his Preface.

[12] In regard to the first, see the literature surveyed in the texts of May (1974), Nowak (2006) and Clark (2010); and in regard to the second see Heal (2000, 2016) and his references. This is no pipe-dream or publication-tactical flourish: given the dissonance between continuous and discrete time, and even though it is a routine matter to obtain the discrete time formulations from their continuous-time counterparts, we have at present no idea of what discrete time results would emerge and how they are to be interpreted in the forms that they take.

[13] Indeed, it is this sentiment that led us to submit this paper in pure theory to the *Journal of Economic Behavior & Organization* in the first place; see Day and Winter Jr. (1980) for the aims of the journal and their repeated use of the word "paradigm".

[14] It is only appropriate to state here that this entire introduction has been written in the light of an insightful referee report – we should like to acknowledge again out gratitude for inspiring this re-writing.

[15] This assumption initially proved contentious. Thus, for example, Hahn (1968) writes that "one is much put off by wide use of a valuation function linear in consumption per head (Shell, Nordhaus, Bruno), although this is sometimes rectified in an appendix. Not only is there nothing to be said for such a valuation, but it sometimes confuses the issue. Thus in the splendid paper by Bruno where the production set is a finite cone, one cannot without additional work be sure whether a particularly odd phase of the optimum trajectory is due to the technology or the silly valuation. The desire for concrete results seems no excuse for asking us to contemplate with equanimity long stretches of time where no one eats at all." Also Intriligator (1968) expresses such reservations. The argument is put most forcefully by Robinson (1969) in the context of the model in Stiglitz (1968, 1970).

An *economy E* consists of a triplet (Ω, u, ρ). A *program from* x_0 is a sequence $\{x_t, y_t\}$ such that for all $t \in \mathbb{N}$, $(x_t, x_{t+1}) \in \Omega$ and $y_t = \max \Lambda(x_t, x_{t+1})$. A program $\{x_t, y_t\}$ is called *stationary* if for all $t \in \mathbb{N}$, $(x_t, y_t) = (x_{t+1}, y_{t+1})$. For all $0 < \rho < 1$, a program $\{x_t^*, y_t^*\}$ from x_0 is said to be *optimal* if

$$\sum_{t=0}^{\infty} \rho^t [u(x_t, x_{t+1}) - u(x_t^*, x_{t+1}^*)] \leq 0$$

for every program $\{x_t, y_t\}$ from x_0.

To rule out the possibility that the long-term capital stock converges to zero under optimal policy, we hereafter impose the following assumption

$$\theta \equiv b/a_I + (1-d) > \rho^{-1}. \tag{2}$$

In words, the assumption[16] ensures the existence of a stock expansible by the factor ρ^{-1}. Denote by ζ the marginal rate of transformation of capital between today and tomorrow under full utilization of both production factors, which is given by

$$\zeta \equiv b/(a_C - a_I) - (1-d) < 0.$$

The modified golden rule stock $\hat{x}$ is defined as a solution to the following constrained optimization problem:

$$u(\hat{x}, \hat{x}) \geq u(x, x') \text{ for all } (x, x') \in \Omega \text{ such that } x \leq (1-\rho)\hat{x} + \rho x'$$

Following the same argument for Proposition 1 in Deng et al. (2019), we can show there exists a modified golden rule stock given by

$$\hat{x} = \frac{a_C(\zeta + 1 - d)}{\zeta + 1} = \frac{a_C b}{b + d(a_C - a_I)},$$

which is the stationary optimal capital stock for the RSL model. Associated with the golden rule stock is the golden rule price $\hat{p} = 1/[(a_C - a_I)(1 + \rho\zeta)]$.

We write explicitly the reduced-form utility function

$$u(x, x') = \begin{cases} \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x', & \text{for } x' \geq \zeta(\hat{x} - x) + \hat{x} \\ \frac{1-d}{b}x - \frac{1}{b}x' + 1, & \text{for } x' < \zeta(\hat{x} - x) + \hat{x} \end{cases}$$

where the first line stands for the case of full utilization of capital while the second line stands for the case of full employment of labor.

We adopt the dynamic programming approach as in Deng et al. (2019) and Deng and Fujio (2020). Define the value function $V : \mathbf{R}_+ \longrightarrow \mathbf{R}$ as

$$V(x) = \sum_{t=0}^{\infty} \rho^t u(x_t, x_{t+1})$$

where $\{x_t, y_t\}$ is an optimal program starting from $x_0 = x$. It can be shown that the value function satisfies the Bellman equation:[17]

$$V(x) = \max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\}$$

for each $x \in \mathbf{R}_+$ with $\Gamma(x) = \{x' : (x, x') \in \Omega\}$. For each $x \in \mathbf{R}_+$, we define the *optimal policy correspondence* $h(x) = \arg\max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\}$. If $h(x)$ is single-valued for any $x \in \mathbf{R}_+$, which is indeed the case in our setting, then we define the *optimal policy function* $g : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ such that $\{g(x)\} = h(x)$ for any $x \in \mathbf{R}_+$. A program $\{x_t, y_t\}$ from x_0 is optimal if and only if it satisfies the equation:[18]

$$V(x_t) = u(x_t, x_{t+1}) + \rho V(x_{t+1}) \text{ for } t \geq 0.$$

Moreover, the value function is continuous, concave, and strictly increasing in x.

3. The optimal policy function

In this section, we first present the main analytical results of the paper to establish the optimality of the straight-down-the-turnpike policy for *any* discount factor that satisfies Assumption (2). We then turn to numerical illustrations which also help form our conjecture of the optimal policy. In the last subsection, we provide a geometric construction of the model, highlighting the intuition behind our main results.

[16] Note this assumption is equivalent to the δ-normality assumption in Khan and Mitra (1986): this has proved to be sufficient for the existence of the modified golden rule stock.

[17] For the existence of a unique solution to the Bellman equation, it suffices to show the Blackwell's sufficient conditions for a contraction (monotonicity and discounting).

[18] This equivalence result is well known in the literature. We refer the interested reader to Footnote 22 in Deng et al. (2019).

3.1. Analytical representation

We postulate the value function based on the straight-down-the-turnpike policy and apply the guess-and-verify method to analytically characterize the optimal policy function.

Theorem 1. *Let $a_C < a_I$ and $0 < d < 1$. The optimal policy function is characterized by the straight-down-the-turnpike policy*

$$g(x) = \begin{cases} \theta x & \text{for } x \in (0, \hat{x}/\theta) \\ \hat{x} & \text{for } x \in [\hat{x}/\theta, \hat{x}/(1-d)) \\ (1-d)x & \text{for } x \in [\hat{x}/(1-d), \infty) \end{cases}$$

According to this theorem,[19] the optimal policy is independent from the discount factor. Under the optimal policy, for any positive initial stock, the economy converges to the golden rule stock in finite periods. Interestingly, unless the economy starts from the golden rule stock, the production factors are never fully utilized along the path of convergence. Theorem 1 can be readily extended to the case of capital intensity being the same in both sectors ($a_C = a_I$), which resembles the one-sector growth model. With $a_C = a_I$, the golden rule stock boils down to be the same as capital intensity in each sector, $\hat{x} = a_I$. By replacing $[\zeta(\hat{x} - x) + \hat{x}]$ with a_I in the proof of Theorem 1, we obtain the following corollary.

Corollary 1. *Let $a_C = a_I$ and $0 < d < 1$. The optimal policy function is characterized by the straight-down-the-turnpike policy*

$$g(x) = \begin{cases} \theta x & \text{for } x \in (0, a_I/\theta) \\ a_I & \text{for } x \in [a_I/\theta, a_I/(1-d)) \\ (1-d)x & \text{for } x \in [a_I/(1-d), \infty) \end{cases}$$

Applying the same argument to the case of circulating capital, we obtain the counterpart of Theorem 1 and Corollary 1 for $d = 1$.[20] Now the straight-down-the-turnpike policy consists of two parts: for the initial stock below $\hat{x}/\theta$, the economy fully specializes in the investment-good sector; for the initial stock above this threshold, the economy produces exactly $\hat{x}$ units of capital.

Theorem 2. *Let $a_C < a_I$ and $d = 1$. The optimal policy function is given by*

$$g(x) = \begin{cases} \theta x & \text{for } x \in (0, \hat{x}/\theta) \\ \hat{x} & \text{for } x \in [\hat{x}/\theta, \infty) \end{cases}$$

Corollary 2. *Let $a_C = a_I$ and $d = 1$. The optimal policy function is given by*

$$g(x) = \begin{cases} \theta x & \text{for } x \in (0, a_I/\theta) \\ a_I & \text{for } x \in [a_I/\theta, \infty) \end{cases}$$

3.2. Numerical representation

To illustrate the theorems and their corollaries, we numerically solve[21] for the optimal policy function by value function iteration for three sets of parameters and report the optimal policies in Fig. 1. In each panel, the optimal policy function is highlighted in red. As suggested by the analytical results, the optimal policy is not sensitive to our specific choice of the discount factor ($1/\theta < \rho < 1$). The first panel plots the case of durable capital ($0 < d < 1$) covered by Theorem 1, the second panel the special case of circulating capital ($d = 1$) covered by Theorem 2, and the third panel another special case of capital intensity being the same in two sectors ($a_I = a_C$) as considered in Corollary 1.

We now turn to a more detailed diagrammatic exploration of the model and its optimal policy.

3.3. Diagrammatic representation

Fig. 2 shows the general feature of the transition possibility set of the model. In this figure, OVL represents the case of the economy fully specializing in the investment-good sector: OV corresponds to a binding capital constraint and VL corresponds to a binding labor constraint. OD represents the case of the economy fully specializing in the consumption-good sector. Both

[19] The proof of Theorem 1 is in the Appendix.

[20] The proof of Theorem 2 is in the Appendix.

[21] Numerical results, which have become part of the standard toolkit and sometimes the core of analysis in other areas of economic growth and macroeconomics at large, have rarely been employed in the work on the two-sector RSS/RSL models of optimal growth. In this connection, we quote from Atkinson (1968), "One disappointing aspect of this book is the absence of any attempt to obtain numerical solutions for the optimal savings programmes. We should be very interested to know what kind of savings rate is implied by the optimal path, and how this is affected by changes in the parameters. Numerical solution would give us considerably more understanding of the form of the optimal policy (particularly when a long way from the steady state) than the decorative but not very informative phase diagrams." Another reviewer of Shell (1967a) asked, " Is it worthwhile restricting oneself, as so many papers in this volume do, to just those simple models which allow one to draw the optimum trajectory? And if so, should we have not been given some guidance how to compute an actual path?" For the role of experimental mathematics, see Bailey et al. (2007) and their references.

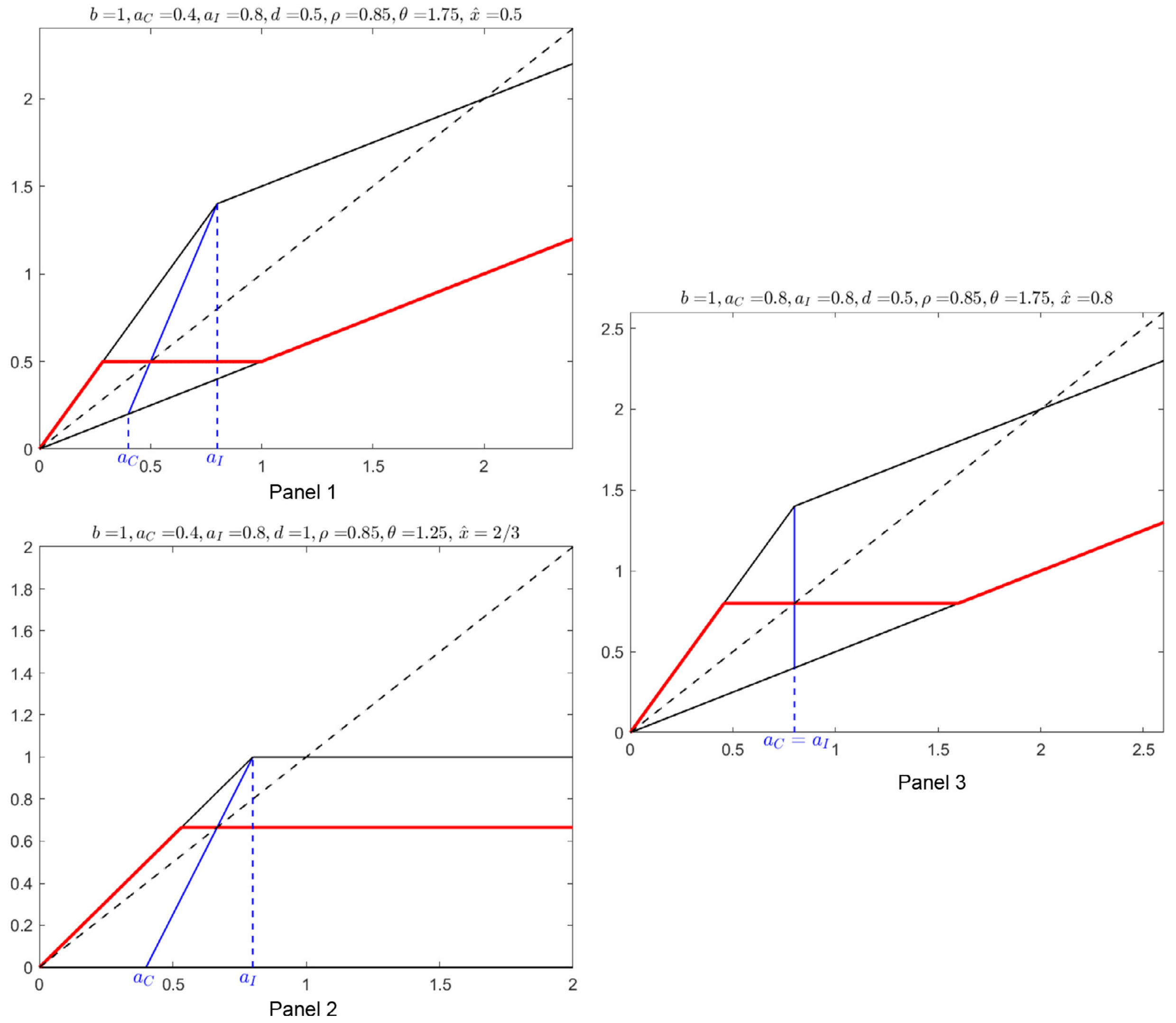

Fig. 1. The Optimal Policy through Numerical Analysis.

capital and labor are fully utilized on MV. There is excess labor for any production plan above MV and excess capital for any plan below MV.

According to the family of indifference curves based on the per-period utility (illustrated as IC_1, IC_2, IC_3), the modified golden rule stock can be found at the point where MV, IC_2, and the geometric signature of the discounted case, the ρ−line ($x' = \frac{1}{\rho}x + \frac{\rho-1}{\rho}\hat{x}$), intersect with each other.[22] The highlighted red line OGG_1D is the straight-down-the-turnpike policy: Under this policy, the economy converges as quickly as possible to the golden rule stock $\hat{x}$.

To gather some further intuition, we define the *value-loss* $\delta^{\rho}_{(\hat{p},\hat{x})}(x, x')$ at the golden-rule price $\hat{p}$ associated with one-period plan (x, x') as

$$\delta^{\rho}_{(\hat{p},\hat{x})}(x, x') \equiv u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} - u(x, x') - \hat{p}(\rho x' - x) \text{ for all } (x, x') \in \Omega.$$

The value loss is minimized and equal to zero for any production plan on MV. According to Lemma 8 in Deng et al. (2019), an optimal program minimizes the discounted sum of the value-losses.

Now consider a special case of $\rho = 1$, which corresponds to the RSL model without discounting as in Fujio (2005). The iso-value-loss lines are parallel to MV. The lines further away from MV are associated with higher value-loss. Without discounting, the period-by-period minimum value loss policy is represented by OM, MV and VL. This distinct (geometric) feature stands in sharp contrast with the case of capital-intensive consumption-good sector ($a_C > a_I$) in which the

[22] The importance of the assumption $\rho > 1/\theta$ can also be seen geometrically. For $\rho < 1/\theta$, to maximize the per-period utility under the constraint of $x \leq (1 - \rho)\hat{x} + \rho x'$, the optimal policy is on the OD line and thus the golden rule stock no longer exists.

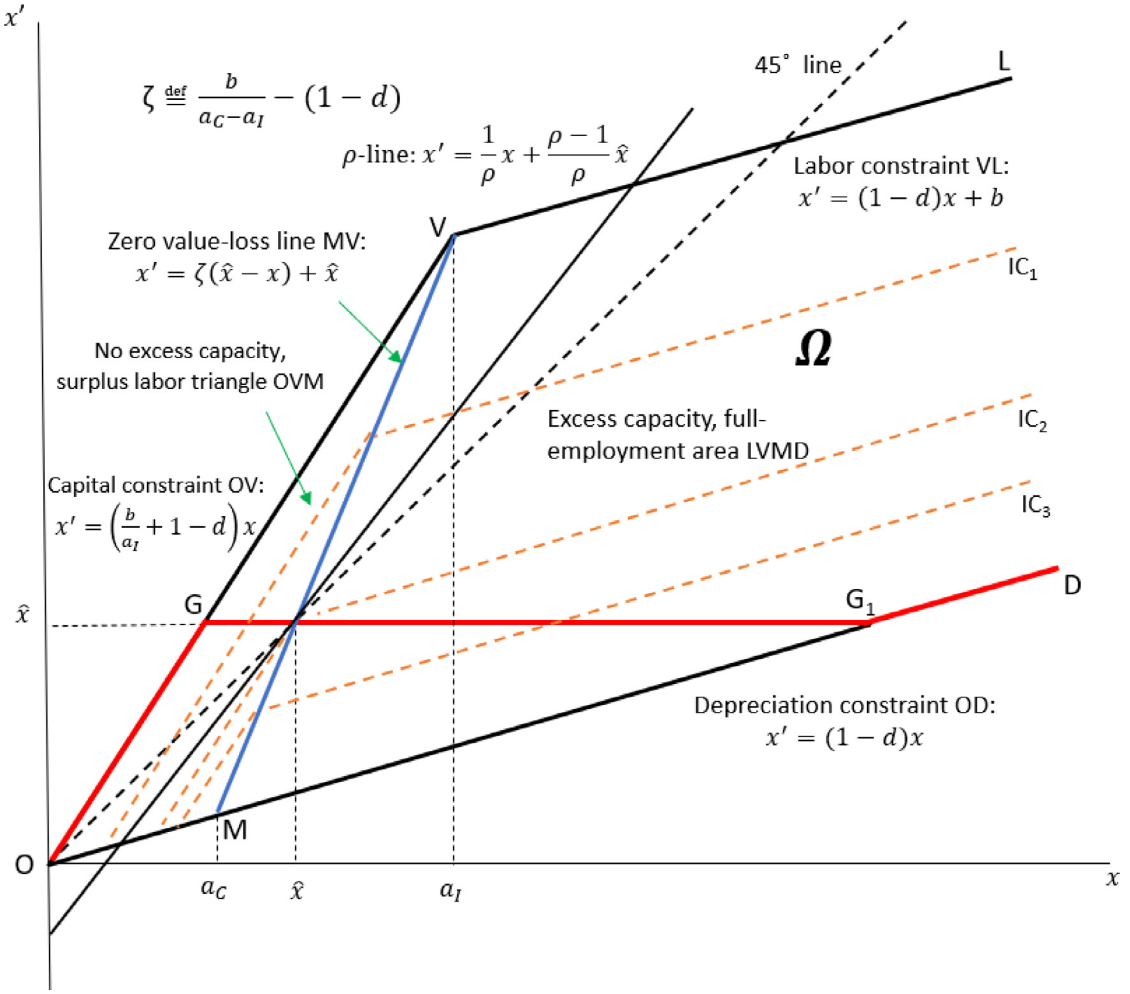

Fig. 2. The Straight-down-the-Turnpike Policy in the RSL Model with $a_I > a_C$.

period-by-period minimum value loss policy is captured by OV, MV and MD. Under this $OM - MV - VL$ policy, the stock will almost surely diverge from the golden rule stock, which cannot be optimal for the model without discounting because the optimal policy must lead to convergence to the golden rule stock in the absence of discounting. In particular, Fujio (2005) demonstrates that, for $\rho = 1$, in order to minimize the total value loss, it is worth taking sacrifice of the value-loss today and adjusting capital stock as quickly as possible to reach the golden rule stock. Therefore, without discounting, the straight-down-the-turnpike policy is optimal.

According to Theorems 1 and 2, the intuition for the undiscounted case extends to the discounted case. In particular, even when ρ is sufficiently smaller than one, which means the agents are more impatient, as long as there still exists a unique golden rule stock, the straight-down-the-turnpike policy remains optimal.

4. A summary and open questions

The significance and the surprise of our characterization results can be clearly seen in the overview of the literature on the RSS and RSL models presented in Table 1 below, the RSS literature stemming from the work on the RSS model originally initiated in 2005 in Khan and Mitra (2005). This work focused on the discrete-time analogue of the Okishio-Robinson-Stiglitz's investigation of the "choice of technique" in development planning and capital theory, and is the special case of the RSL model with $a_I = 0$. The uniform results reported in this paper bring out sharply that there is simply no necessity for any entry regarding the case $a_C \leq a_I$ in Table 1.

In a recent demonstration, Deng and Fujio (2020) show that the straight-down-the-turnpike policy and its variants can be optimal for the case of a capital-intensive consumption-good sector provided that ζ is greater than one and the discount factor is sufficiently high, $\rho > 1/\zeta$. The question then naturally arises as to what happens in the (sufficiently impatient) case of the discount factor being below $1/\zeta$. Moreover, the existing work on the RSL model has always imposed the assumption

Table 1
The RSS and RSL Models: An Overview of the Literature.[23]

The Specific RSS Setting (Khan-Mitra): $a_C > a_I = 0$		
$\xi \le 1$	$\xi < 0$	Monotonic convergence
	$\xi = 0$	Straight-down-the-turnpike
	$0 < \xi < 1$	Dampened convergence
	$\xi = 1$	Two-period cycles
$\xi > 1$	$\xi > 1/\rho$	Neighborhood turnpike theorem
	$\xi \le 1/\rho$	Complicated & chaotic dynamics
The General RSL Setting: $a_C > a_I$		
$\zeta \le 1$ (DFK19)	$\zeta < 0$	Monotonic convergence
	$\zeta = 0$	Straight-down-the-turnpike
	$0 < \zeta < 1$	Dampened convergence
	$\zeta = 1$	Two-period cycles
$\zeta > 1$	$\zeta > 1/\rho$ (DF20)	Neighborhood turnpike theorem
	$\zeta \le 1/\rho$ (Open)	Complicated & chaotic dynamics

of $\rho > 1/\theta$ to ensure the existence of the modified golden rule stock. What about optimal policy for $\rho \le 1/\theta$? Does the optimal policy always lead to extinction? We leave these open questions for future research.

Finally, we draw yet again the reader's attention to our intended applications in bioeconomics and evolutionary dynamics, as laid out in the penultimate paragraph of the introduction.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Proof of Theorem 1. Postulate a candidate value function based on the straight-down-the-turnpike policy

$$W(x) = \begin{cases} \frac{a_I\theta}{a_C b}\rho^n(\theta^n x - \hat{x}) + \frac{\rho^n}{1-\rho}u(\hat{x},\hat{x}) & \text{for } x \in [\frac{\hat{x}}{\theta^{n+1}}, \frac{\hat{x}}{\theta^n}) \\ \frac{1-d}{b}\rho^n\left[(1-d)^n x - \hat{x}\right] + \frac{1-\rho^n+\rho^n u(\hat{x},\hat{x})}{1-\rho} & \text{for } x \in [\frac{\hat{x}}{(1-d)^n}, \frac{\hat{x}}{(1-d)^{n+1}}) \end{cases}$$

where $n = 0, 1, 2\ldots$ We now verify if $W(x)$ satisfies the Bellman equation

$$W(x) = \max_{x' \in \Gamma(x)} \{u(x, x') + \rho W(x')\}$$

We consider four cases: (1) $x \in [\hat{x}/\theta, \hat{x})$; (2) $x \in [\hat{x}, \frac{\hat{x}}{1-d})$; (3) $x \in [\frac{\hat{x}}{\theta^{n+1}}, \frac{\hat{x}}{\theta^n})$ with $n \ge 1$; (4) $x \in [\frac{\hat{x}}{(1-d)^n}, \frac{\hat{x}}{(1-d)^{n+1}})$ with $n \ge 1$.

Case (1): $x \in [\hat{x}/\theta, \hat{x})$.

For $x \in [\hat{x}/\theta, \hat{x})$, $W(x) = \frac{a_I\theta}{a_C b}(x - \hat{x}) + \frac{1}{1-\rho}u(\hat{x},\hat{x}) = (\frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}\hat{x}) + \frac{\rho}{1-\rho}u(\hat{x},\hat{x}) = u(x,\hat{x}) + \rho W(\hat{x})$, where the second equality follows from $u(\hat{x},\hat{x}) = \frac{a_I}{a_C b}(\theta - 1)\hat{x}$ and the last equality follows from $u(x,\hat{x}) = \frac{a_I}{a_C b}(\theta x - \hat{x})$ for $x \in [\hat{x}/\theta, \hat{x})$ and $W(\hat{x}) = \frac{u(\hat{x},\hat{x})}{1-\rho}$. Pick $x' > \hat{x}$ such that $(x, x') \in \Omega$. There exists $n_0 \ge 0$ such that $x' \in [\frac{\hat{x}}{(1-d)^{n_0}}, \frac{\hat{x}}{(1-d)^{n_0+1}})$. Since $x' \ge \zeta(\hat{x} - x) + \hat{x}$, we have

$$\begin{aligned} W_0(x') &\equiv u(x, x') + \rho W(x') \\ &= \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' + \frac{1-d}{b}\rho^{n_0+1}\left[(1-d)^{n_0}x' - \hat{x}\right] + \frac{\rho - \rho^{n_0+1} + \rho^{n_0+1}u(\hat{x},\hat{x})}{1-\rho}; \\ \frac{\partial W_0(x')}{\partial x'} &= \frac{1}{a_C b}\left[a_C\rho^{n_0+1}(1-d)^{n_0+1} - a_I\right] < 0, \end{aligned}$$

where the inequality follows from $a_C < a_I$, $\rho < 1$, and $(1-d) < 1$. Pick x' such that $\hat{x} > x' \ge \zeta(\hat{x} - x) + \hat{x}$. There exists $n_0 \ge 0$ such that $x' \in [\frac{\hat{x}}{\theta^{n_0+1}}, \frac{\hat{x}}{\theta^{n_0}})$. Since $x' \ge \zeta(\hat{x} - x) + \hat{x}$, we have

$$\begin{aligned} W_0(x) &\equiv u(x, x') + \rho W(x') = \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' + \frac{a_I\theta}{a_C b}\rho^{n_0+1}(\theta^{n_0}x' - \hat{x}) + \frac{\rho^{n_0+1}}{1-\rho}u(\hat{x},\hat{x}); \\ \frac{\partial W_0(x')}{\partial x'} &= \frac{a_I}{a_C b}\left[(\rho\theta)^{n_0+1} - 1\right] > 0, \end{aligned}$$

where the inequality follows from $\rho\theta > 1$. Finally, pick $x' < \zeta(\hat{x} - x) + \hat{x} < \hat{x}$ such that $(x, x') \in \Omega$. There exists $n_0 \ge 0$ such that $x' \in [\frac{\hat{x}}{\theta^{n_0+1}}, \frac{\hat{x}}{\theta^{n_0}})$. Since $x' < \zeta(\hat{x} - x) + \hat{x}$, we have

$$W_0(x) \equiv u(x, x') + \rho W(x') = \frac{1-d}{b}x - \frac{1}{b}x' + 1 + \frac{a_I\theta}{a_C b}\rho^{n_0+1}(\theta^{n_0}x' - \hat{x}) + \frac{\rho^{n_0+1}}{1-\rho}u(\hat{x},\hat{x});$$

$$\frac{\partial W_0(x')}{\partial x'} = \frac{1}{b}\left[\frac{a_I}{a_C}(\rho\theta)^{n_0+1} - 1\right] > 0,$$

where the inequality follows from $\rho\theta > 1$ and $a_I > a_C$. Taken together, we have shown that $W_0(x')$ strictly decreases with x' for $x' > \hat{x}$ and strictly increases with x' for $x' < \hat{x}$, so $W_0(\cdot)$ is maximized when $x' = \hat{x}$.

Case (2): $x \in [\hat{x}, \frac{\hat{x}}{1-d})$.

For $x \in [\hat{x}, \frac{\hat{x}}{1-d})$, $W(x) = \frac{1-d}{b}\left(x - \hat{x}\right) + \frac{1}{1-\rho}u(\hat{x},\hat{x}) = \frac{1-d}{b}x - \frac{1}{b}\hat{x} + 1 + \frac{\rho}{1-\rho}u(\hat{x},\hat{x}) = u(x,\hat{x}) + \rho W(\hat{x})$, where the second equality follows from $u(\hat{x},\hat{x}) = 1 - \frac{d}{b}\hat{x}$ and the last equality follows from $u(x,\hat{x}) = \frac{1-d}{b}x - \frac{1}{b}\hat{x} + 1$ for $x \in [\hat{x}, \frac{\hat{x}}{1-d})$ and $W(\hat{x}) = \frac{u(\hat{x},\hat{x})}{1-\rho}$. Pick x' such that $\zeta(\hat{x} - x) + \hat{x} > x' > \hat{x}$. There exists $n_0 \geq 0$ such that $x' \in [\frac{\hat{x}}{(1-d)^{n_0}}, \frac{\hat{x}}{(1-d)^{n_0+1}})$. Since $x' < \zeta(\hat{x} - x) + \hat{x}$, we have

$$\begin{aligned} W_0(x') &\equiv u(x,x') + \rho W(x') \\ &= \frac{1-d}{b}x - \frac{1}{b}x' + 1 + \frac{1-d}{b}\rho^{n_0+1}\left[(1-d)^{n_0}x' - \hat{x}\right] + \frac{\rho - \rho^{n_0+1} + \rho^{n_0+1}u(\hat{x},\hat{x})}{1-\rho}; \\ \frac{\partial W_0(x')}{\partial x'} &= \frac{1}{b}\left[\rho^{n_0+1}(1-d)^{n_0+1} - 1\right] < 0, \end{aligned}$$

where the inequality follows from $\rho < 1$ and $(1-d) < 1$. Pick $x' > \zeta(\hat{x} - x) + \hat{x} \geq \hat{x}$ such that $(x,x') \in \Omega$. There exists $n_0 \geq 0$ such that $x' \in [\frac{\hat{x}}{(1-d)^{n_0}}, \frac{\hat{x}}{(1-d)^{n_0+1}})$. Since $x' \geq \zeta(\hat{x} - x) + \hat{x}$, we have

$$\begin{aligned} W_0(x') &\equiv u(x,x') + \rho W(x') \\ &= \frac{a_I\theta}{a_C b}x - \frac{a_I}{a_C b}x' + \frac{1-d}{b}\rho^{n_0+1}\left[(1-d)^{n_0}x' - \hat{x}\right] + \frac{\rho - \rho^{n_0+1} + \rho^{n_0+1}u(\hat{x},\hat{x})}{1-\rho}; \\ \frac{\partial W_0(x')}{\partial x'} &= \frac{1}{a_C b}\left[a_C\rho^{n_0+1}(1-d)^{n_0+1} - a_I\right] < 0, \end{aligned}$$

where the inequality follows from $a_C < a_I$, $\rho < 1$, and $(1-d) < 1$. Finally, pick $x' < \hat{x} \leq \zeta(\hat{x} - x) + \hat{x}$ such that $(x,x') \in \Omega$. There exists $n_0 \geq 0$ such that $x' \in [\frac{\hat{x}}{\theta^{n_0+1}}, \frac{\hat{x}}{\theta^{n_0}})$. Since $x' < \zeta(\hat{x} - x) + \hat{x}$, we have

$$W_0(x) \equiv u(x,x') + \rho W(x') = \frac{1-d}{b}x - \frac{1}{b}x' + 1 + \frac{a_I\theta}{a_C b}\rho^{n_0+1}(\theta^{n_0}x' - \hat{x}) + \frac{\rho^{n_0+1}}{1-\rho}u(\hat{x},\hat{x});$$

$$\frac{\partial W_0(x')}{\partial x'} = \frac{1}{b}\left[\frac{a_I}{a_C}(\rho\theta)^{n_0+1} - 1\right] > 0,$$

where the inequality follows from $\rho\theta > 1$ and $a_I > a_C$. Taken together, we have shown that $W_0(x')$ strictly decreases with x' for $x' > \hat{x}$ and strictly increases with x' for $x' < \hat{x}$, so $W_0(\cdot)$ is maximized when $x' = \hat{x}$.

Case (3): $x \in [\frac{\hat{x}}{\theta^{n+1}}, \frac{\hat{x}}{\theta^n})$ with $n \geq 1$

For $x \in [\frac{\hat{x}}{\theta^{n+1}}, \frac{\hat{x}}{\theta^n})$ with $n \geq 1$, $W(x) = \rho(\frac{a_I\theta}{a_C b}\rho^{n-1}(\theta^{n-1}(\theta x) - \hat{x}) + \frac{\rho^{n-1}}{1-\rho}u(\hat{x},\hat{x})) = u(x,\theta x) + \rho W(\theta x)$, where the second equality follows from $u(x,\theta x) = 0$ and $\theta x \in [\frac{\hat{x}}{\theta^n}, \frac{\hat{x}}{\theta^{n-1}})$. For $x' < \theta x$, we can follow essentially the same argument as that for the case of $x \in [\frac{\hat{x}}{\theta}, \hat{x})$ to show that $W_0(x)$ strictly increases with x', so $W_0(\cdot)$ is maximized when $x' = \theta x$.

Case (4): $x \in [\frac{\hat{x}}{(1-d)^n}, \frac{\hat{x}}{(1-d)^{n+1}})$ with $n \geq 1$.

For $x \in [\frac{\hat{x}}{(1-d)^n}, \frac{\hat{x}}{(1-d)^{n+1}})$ with $n \geq 1$, $W(x) = \frac{1-d}{b}\rho^n[(1-d)^n x - \hat{x}] + \frac{1-\rho^n+\rho^n u(\hat{x},\hat{x})}{1-\rho} = 1 + \rho(\frac{1-d}{b}\rho^{n-1}[(1-d)^{n-1}(1-d)x - \hat{x}] + \frac{1-\rho^{n-1}+\rho^{n-1}u(\hat{x},\hat{x})}{1-\rho}) = u(x,(1-d)x) + \rho W((1-d)x)$, where the last equality follows from $u(x,(1-d)x) = 1$ and $(1-d)x \in [\frac{\hat{x}}{(1-d)^{n-1}}, \frac{\hat{x}}{(1-d)^n})$. For $x' > (1-d)x$, we can follow essentially the same argument as that for the case of $x \in [\hat{x}, \frac{\hat{x}}{1-d})$ to show that $W_0(x)$ strictly decreases with x', so $W_0(\cdot)$ is maximized when $x' = (1-d)x$.

In sum, we have verified that the postulated value function W satisfies the Bellman equation and the straight-down-the-turnpike policy is optimal. □

Proof of Theorem 2. The proof essentially follows the proof of Theorem 1. Since $d = 1$, $\theta = \frac{a_I}{b}$. Postulate a candidate value function based on the straight-down-the-turnpike policy

$$W(x) = \begin{cases} \frac{1}{a_C}\rho^n(\theta^n x - \hat{x}) + \frac{\rho^n}{1-\rho}u(\hat{x},\hat{x}) & \text{for } x \in [\frac{\hat{x}}{\theta^{n+1}}, \frac{\hat{x}}{\theta^n}) \\ \frac{1}{1-\rho}u(\hat{x},\hat{x}) & \text{for } x \in [\hat{x}, \infty) \end{cases}$$

where $n = 0, 1, 2...$ Following the same steps as in the proof of Theorem 1, we can verify that $W(x)$ satisfies the Bellman equation

$$W(x) = \max_{x' \in \Gamma(x)} \{u(x,x') + \rho W(x')\}$$

for each of the three cases: (1) $x \in [\hat{x}/\theta, \hat{x})$; (2) $x \in [\hat{x}, \infty)$; (3) $x \in [\frac{\hat{x}}{\theta^{n+1}}, \frac{\hat{x}}{\theta^n})$ with $n \geq 1$. We thus obtain the desired conclusion. □

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