Online Appendix of
"Optimal Growth in the Robinson-Shinkai-Leontief Model: The Case of Capital-Intensive Consumption Goods"

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1 Appendix: Proofs of the Results

Proof of Proposition 1:

Since $a_C > a_I$, $\zeta > d-1$, which implies $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$. Given Condition 3, $b/a_I > d + 1/\rho - 1 > d$, where the second inequality stems from $0 < \rho < 1$. Since $b/a_I > d$, $d\hat{x} = dba_C/(da_C + b - da_I) < b$ and $d\hat{x} < b\hat{x}/a_I$. Therefore, $d\hat{x} < b\min\{1, \hat{x}/a_I\}$. Then we must have $(\hat{x}, \hat{x}) \in \Omega$.

Consider y in $\Lambda(x, x')$. Define

$$\begin{aligned}
\alpha(x, x', y) &= (1/a_C)(x - (a_I/b)(x' - (1-d)x)) - y, \\
\beta(x, x', y) &= 1 - (1/b)(x' - (1-d)x) - y.
\end{aligned}$$

We have

$$y + \hat{p}(\rho x' - x) = (1-A) - A\alpha(x, x', y) - (1-A)\beta(x, x', y),$$

where $A \equiv \frac{a_C(1-\rho(1-d))}{(a_C - a_I)(1+\rho\zeta)}$. Since $a_C > a_I$ (Condition 1), $\zeta > -1$, and we know $0 < \rho < 1$ and $0 < d < 1$, we have $A > 0$. Given Condition 3, $A < 1$. By construction, $\alpha(x, x', y) \geq 0$ and $\beta(x, x', y) \geq 0$, which implies

$$y + \hat{p}(\rho x' - x) \leq 1 - A.$$

Let $\hat{y} = 1 - (d/b)\hat{x} = (1/a_C)(1 - a_I d/b)\hat{x} = (b - da_I)/(b - da_I + da_C) > 0$. We have $\hat{y} \in \Lambda(\hat{x}, \hat{x})$ and $\alpha(\hat{x}, \hat{x}, \hat{y}) = \beta(\hat{x}, \hat{x}, \hat{y}) = 0$. Therefore, $\hat{y} = u(\hat{x}, \hat{x})$ and $\hat{y} + (\rho - 1)\hat{p}\hat{x} = 1 - A$, which implies

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq y + \hat{p}(\rho x' - x) \text{ for all } (x, x') \in \Omega \text{ and } y \in \Lambda(x, x'). \tag{1}$$

Since $u(x, x') = \max \Lambda(x, x')$, we then obtain the desired inequality

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x) \text{ for all } (x, x') \in \Omega. \tag{2}$$

Proof of Lemma 1:

When labor and capital are fully utilized, we must have

$$(1/a_C)(x - (a_I/b)(x' - (1-d)x)) = 1 - (1/b)(x' - (1-d)x).$$

Let $y = (1/a_C)(x - (a_I/b)(x' - (1-d)x)) = 1 - (1/b)(x' - (1-d)x)$. Following the argument in the proof of Proposition 1, we then have $\alpha(x, x', y) = \beta(x, x', y) = 0$, and we have shown $\alpha(\hat{x}, \hat{x}, \hat{y}) = \beta(\hat{x}, \hat{x}, \hat{y}) = 0$, so $u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} = u(x, x') + \hat{p}(\rho x' - x)$, or equivalently, $\delta^\rho(x, x') = 0$.

Proof of Lemma 2:

Given Condition 3, we have $\rho b > a_I - \rho(1-d)a_I$, and since we know $a_C > a_I$, we must have

$$\frac{\rho}{(a_C - a_I) + \rho b - \rho(1-d)(a_C - a_I)} > \frac{a_I}{a_C b},$$

where the left hand side is equal to $\hat{p}\rho$.

Moreover, since $a_C > a_I$, $\rho \in (0,1)$ and $d \in (0,1)$, $(a_C - a_I)(1 - \rho(1-d)) > 0$, which implies

$$\hat{p}\rho = \frac{\rho}{(a_C - a_I) + \rho b - \rho(1-d)(a_C - a_I)} < \frac{1}{b}.$$

Proof of Lemma 3:

For (i), consider x_1 and x_2 in X and λ in $(0,1)$. Let $x_3 = \lambda x_1 + (1-\lambda)x_2$. Let $\{x_i(t), y_i(t)\}$ be an optimal program starting from x_i for $i = 1,2,3$. By construction, Ω is convex. To see u being concave, letting $x = \lambda x_1(t) + (1-\lambda)x_2(t)$ and $x' = \lambda x_1(t+1) + (1-\lambda)x_2(t+1)$, we have

$$\begin{aligned}
&u(x,x') = \min\{(1/a_C)(x - (a_I/b)(x' - (1-d)x)), 1 - (1/b)(x' - (1-d)x)\} \\
&\geq \lambda \min\{(1/a_C)(x_1(t) - (a_I/b)(x_1(t+1) - (1-d)x_1(t))), 1 - (1/b)(x_1(t+1) - (1-d)x_1(t))\} \\
&+(1-\lambda)\min\{(1/a_C)(x_2(t) - (a_I/b)(x_2(t+1) - (1-d)x_2(t))), 1 - (1/b)(x_2(t+1) - (1-d)x_2(t))\} \\
&= \lambda u(x_1(t), x_1(t+1)) + (1-\lambda)u(x_2(t), x_2(t+1)).
\end{aligned}$$

Since u is concave, we have

$$\begin{aligned}
\lambda V(x_1) + (1-\lambda)V(x_2) &\leq \sum_{t=0}^{\infty} \rho^t u(\lambda x_1(t) + (1-\lambda)x_2(t), \lambda x_1(t+1) + (1-\lambda)x_2(t+1)) \\
&\leq V(\lambda x_1 + (1-\lambda)x_2),
\end{aligned}$$

where the second inequality follows from the fact that $\{\lambda x_1(t) + (1-\lambda)x_2(t)\}_{t=0}^{\infty}$ generates a program starting from $x_3 = \lambda x_1 + (1-\lambda)x_2$.

For (ii), consider two initial stocks, x_1 and x_2 with $x_1 < x_2$. Let $x_1' \in \arg\max_{x' \in \Gamma(x_1)}\{u(x_1, x') + \rho V(x')\}$. We then have $V(x_1) = u(x_1, x_1') + \rho V(x_1')$. By the optimality of V, $V(x_2) \geq u(x_2, x_1') + \rho V(x_1') > u(x_1, x_1') + \rho V(x_1') = V(x_1)$, where the second inequality follows from $u(x,x')$ being strictly increasing with x.

Proof of Lemma 4:

For i), based on the definition of the value function,

$$\begin{aligned}
V(x) - V(\hat{x}) &= \sum_{t=0}^{\infty} \rho^t [u(x(t), x(t+1)) - u(\hat{x}, \hat{x})] \\
&\leq \sum_{t=0}^{\infty} \rho^t [(\rho - 1)\hat{p}\hat{x} - \hat{p}(\rho x(t+1) - x(t))] \\
&= (\rho - 1)\hat{p}\hat{x}/(1-\rho) + \hat{p}x(0) = \hat{p}(x - \hat{x})
\end{aligned}$$

where $x = x(0)$ and the inequality follows from Equation 4.

For ii), take x such as $x = \hat{x} + \epsilon$ for $\epsilon > 0$. Letting $\epsilon \to 0$, we obtain $V'_+(\hat{x}) \leq \hat{p}$. Similarly, take x such as $x = \hat{x} - \epsilon$ for $\epsilon > 0$. Letting $\epsilon \to 0$, we obtain $\hat{p} \leq V'_-(\hat{x})$.

Proof of Lemma 5:

Condition 1 implies $\zeta > d - 1$, which guarantees $\hat{x} = a_C(\zeta + 1 - d)/(\zeta + 1) < a_C$.

Condition 3 implies $b > da_I$ and we know $a_C > a_I$, so we have

$$\hat{x} - a_I = \frac{a_C b}{a_C d - a_I d + b} - a_I = \frac{(a_C - a_I)(b - da_I)}{a_C d - a_I d + b} > 0.$$

Therefore, $\hat{x} > a_I$.

So far we have shown that $a_I < \hat{x} < a_C$ holds for any ζ.

We now consider the relationship between $\hat{x}/(1-d)$ and a_C. Since $\hat{x}/(1-d) = a_C(\zeta+1-d)/(\zeta+1-d-d\zeta)$, $\hat{x}/(1-d) > a_C$. if and only if $\zeta > 0$. If $\zeta = 0$, $\hat{x}/(1-d) = a_C$. If $\zeta < 0$, $\hat{x}/(1-d) < a_C$, but $\hat{x}/(1-d) > \hat{x}$ still holds because $d > 0$.

Last, we consider the relationship between $\hat{x}/\theta$ and a_I.

$$\begin{aligned}
\theta a_I - \hat{x} &= b + (1-d)a_I - \frac{a_C(\zeta + 1 - d)}{\zeta + 1} \\
&= \zeta(a_C - a_I) + (1-d)a_C - \frac{a_C(\zeta + 1 - d)}{\zeta + 1} \\
&= \zeta\left(a_C - a_I - \frac{a_C d}{\zeta + 1}\right) \\
&= \frac{\zeta}{\zeta + 1}\left(b + d(a_C - a_I) - a_C d\right) \\
&= \frac{\zeta}{\zeta + 1}\left(b - da_I\right).
\end{aligned}$$

Hence, $a_I > \hat{x}/\theta$ if and only if $\zeta > 0$. If $\zeta = 0$, $a_I = \hat{x}/\theta$. If $\zeta < 0$, $a_I < \hat{x}/\theta$, but $\hat{x}/\theta < \hat{x}$ still holds because $\theta > 1$ according to Condition 3.

Proof of Lemma 6:

Suppose on the contrary there exists $x \in (0, \hat{x}/\theta]$ and $z \in h(x)$ such that $z \neq \theta x$. Since $\zeta > 0$, according to Lemma 5, $x \leq \hat{x}/\theta < a_I$. Then we must have $z < \theta x$, and

$$V(x) = u(x, z) + \rho V(z) \geq u(x, \theta x) + \rho V(\theta x).$$

Rearranging the equation, we have

$$u(x,z) - u(x,\theta x) \geq \rho(V(\theta x) - V(z)) \geq \rho V'_-(\theta x)(\theta x - z) \geq \rho V'_-(\hat{x})(\theta x - z) \geq \rho\hat{p}(\theta x - z),$$

where the second inequality follows from concavity of V, the third inequality follows from concavity of V and the fact that $\theta x \leq \hat{x}$ for $x \in (0, \hat{x}/\theta]$, and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (0, \hat{x}/\theta]$, we have

$$u(x,z) - u(x,\theta x) = (1/a_C)[x - (a_I/b)(z-(1-d)x)] = \frac{a_I}{ba_C}(\theta x - z) < \hat{p}\rho(\theta x - z),$$

where the last inequality follows from Lemma 2. Since we have shown that $u(x,z) - u(x,\theta x) \geq \rho\hat{p}(\theta x - z)$. This leads to a contradiction and establishes the desired result.
Proof of Lemma 7:

Suppose on the contrary there exists $x \in [\hat{x}/(1-d), \infty)$ and $z \in h(x)$ such that $z \neq (1-d)x$. Then we must have $z > (1-d)x$, and

$$V(x) = u(x,z) + \rho V(z) \geq u(x,(1-d)x) + \rho V((1-d)x).$$

Rearranging the equation, we have

$$\begin{aligned} u(x,(1-d)x) - u(x,z) &\leq \rho(V(z) - V((1-d)x)) \leq \rho V'_+((1-d)x)(z-(1-d)x) \\ &\leq \rho V'_+(\hat{x})(z-(1-d)x) \leq \rho\hat{p}(z-(1-d)x), \end{aligned}$$

where the second inequality follows from concavity of V, the third inequality follows from concavity of V and the fact that $\hat{x} \leq (1-d)x$, and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in [\hat{x}/(1-d), \infty)$, we have

$$u(x,(1-d)x) - u(x,z) = (z-(1-d)x)/b > \hat{p}\rho(z-(1-d)x),$$

where the last inequality follows from Lemma 2. Since we have shown that $u(x,(1-d)x) - u(x,z) \leq \rho\hat{p}(z-(1-d)x)$. This leads to a contradiction and establishes the desired result.
Proof of Proposition 2:

We proceed by going over each subregion.

Subregion $(\hat{x}/\theta, a_I]$:

Suppose on the contrary there exists $x \in (\hat{x}/\theta, a_I]$ and $z \in h(x)$ such that $z \notin [\hat{x}, \theta x]$. Then we must have $z < \hat{x}$. By the optimality of z,

$$V(x) = u(x,z) + \rho V(z) \geq u(x,\hat{x}) + \rho V(\hat{x}).$$

Rearranging the equation, we have

$$u(x,z) - u(x,\hat{x}) \geq \rho(V(\hat{x}) - V(z)) \geq \rho V'_-(\hat{x})(\hat{x} - z) \geq \rho\hat{p}(\hat{x} - z),$$

where the second inequality follows from concavity of V and the third inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (\hat{x}/\theta, a_I]$, we have

$$u(x,z) - u(x,\hat{x}) \leq \frac{a_I}{ba_C}(\hat{x} - z) < \hat{p}\rho(\hat{x} - z),$$

where the last inequality follows from Lemma 2.

Since we have shown that $u(x,z) - u(x,\hat{x}) \geq \rho\hat{p}(\hat{x} - z)$. This leads to a contradiction and establishes the desired result.

Subregion $(a_I, \hat{x}]$**:**

Consider $(a_I, \hat{x}]$.[4] Suppose on the contrary there exists $x \in (a_I, \hat{x}]$ and $z \in h(x)$ such that $z \notin [\hat{x}, \zeta(\hat{x}-x)+\hat{x}]$. There are two possible cases: (i) $z < \hat{x}$; (ii) $z > \zeta(\hat{x}-x)+\hat{x}$.

Consider (i) $z < \hat{x}$. We have

$$V(x) = u(x,z) + \rho V(z) \geq u(x,\hat{x}) + \rho V(\hat{x}).$$

Rearranging the equation, we have

$$u(x,z) - u(x,\hat{x}) \geq \rho(V(\hat{x}) - V(z)) \geq \rho V'_{-}(\hat{x})(\hat{x} - z) \geq \rho\hat{p}(\hat{x} - z),$$

where the second inequality follows from concavity of V and the third inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (a_I, \hat{x}]$, we have

$$u(x,z) - u(x,\hat{x}) \leq \frac{a_I}{ba_C}(\hat{x} - z) < \hat{p}\rho(\hat{x} - z),$$

where the last inequality follows from Lemma 2.

Since we have shown that $u(x,z) - u(x,\hat{x}) \geq \rho\hat{p}(\hat{x} - z)$. This leads to a contradiction and establishes the desired result.

Consider (ii) $z > \zeta(\hat{x} - x) + \hat{x}$. We have

$$V(x) = u(x,z) + \rho V(z) \geq u(x, \zeta(\hat{x} - x) + \hat{x}) + \rho V(\zeta(\hat{x} - x) + \hat{x}).$$

Rearranging the equation, we have

$$\begin{aligned} u(x, \zeta(\hat{x} - x) + \hat{x}) - u(x,z) &\leq \rho(V(z) - V(\zeta(\hat{x} - x) + \hat{x})) \\ &\leq \rho V'_{+}(\zeta(\hat{x} - x) + \hat{x})(z - (\zeta(\hat{x} - x) + \hat{x})) \\ &\leq \rho V'_{+}(\hat{x})(z - (\zeta(\hat{x} - x) + \hat{x})) \leq \rho\hat{p}(z - (\zeta(\hat{x} - x) + \hat{x})) \end{aligned}$$

[4]This is the range of $(0, \hat{x}]$ in the RSS model. When $a_I = 0$, the optimal policy correspondence can be further reduced to a function. For a complete characterization for $x \in (a_I, \hat{x}]$ in the RSS model, see Lemma 2 in [1].

where the second inequality follows from concavity of V, the third inequality follows from concavity of V and the fact that $\zeta(\hat{x} - x) + \hat{x} \geq \hat{x}$ for $x \in (a_I, \hat{x}]$, and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (a_I, \hat{x}]$, we have

$$u(x, \zeta(\hat{x} - x) + \hat{x}) - u(x, z) \geq (1/b)(z - (\zeta(\hat{x} - x) + \hat{x})) > \hat{p}\rho(z - (\zeta(\hat{x} - x) + \hat{x})),$$

where the last inequality follows from Lemma 2.

Since we have shown that $u(x, (\zeta(\hat{x} - x) + \hat{x})) - u(x, z) \leq \rho\hat{p}(z - (\zeta(\hat{x} - x) + \hat{x}))$. This leads to a contradiction and establishes the desired result.

Subregion $(\hat{x}, a_C]$**:**

Suppose on the contrary there exists $x \in (\hat{x}, a_C]$ and $z \in h(x)$ such that $z \notin [\zeta(\hat{x} - x) + \hat{x}, \hat{x}]$. There are two possible cases: (i) $z > \hat{x}$; (ii) $z < \zeta(\hat{x} - x) + \hat{x}$.

Consider (i) $z > \hat{x}$. We have

$$V(x) = u(x, z) + \rho V(z) \geq u(x, \hat{x}) + \rho V(\hat{x}).$$

Rearranging the equation, we have

$$u(x, \hat{x}) - u(x, z) \leq \rho(V(z) - V(\hat{x})) \leq \rho V_+'(\hat{x})(z - \hat{x}) \leq \rho\hat{p}(z - \hat{x}),$$

where the second inequality follows from concavity of V and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (\hat{x}, a_C]$, we have

$$u(x, \hat{x}) - u(x, z) \geq (1/b)(z - \hat{x}) > \hat{p}\rho(z - \hat{x}),$$

where the last inequality follows from Lemma 2.

Since we have shown that $u(x, \hat{x}) - u(x, z) \leq \rho\hat{p}(z - \hat{x})$. This leads to a contradiction and establishes the desired result.

Consider (ii) $z < \zeta(\hat{x} - x) + \hat{x}$. We have

$$V(x) = u(x, z) + \rho V(z) \geq u(x, \zeta(\hat{x} - x) + \hat{x}) + \rho V(\zeta(\hat{x} - x) + \hat{x}).$$

Rearranging the equation, we have

$$\begin{aligned} u(x, z) - u(x, \zeta(\hat{x} - x) + \hat{x}) &\geq \rho(V(\zeta(\hat{x} - x) + \hat{x}) - V(z)) \\ &\geq \rho V_-'(\zeta(\hat{x} - x) + \hat{x})((\zeta(\hat{x} - x) + \hat{x}) - z) \\ &\geq \rho V_-'(\hat{x})((\zeta(\hat{x} - x) + \hat{x}) - z) \geq \rho\hat{p}((\zeta(\hat{x} - x) + \hat{x}) - z) \end{aligned}$$

where the second inequality follows from concavity of V, the third inequality follows from concavity of V and the fact that $\zeta(\hat{x} - x) + \hat{x} \leq \hat{x}$ for $x \in (\hat{x}, a_C]$, and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (\hat{x}, a_C]$, we have

$$u(x,z) - u(x, \zeta(\hat{x}-x)+\hat{x}) \leq \frac{a_I}{a_C b}((\zeta(\hat{x}-x)+\hat{x})-z) < \hat{p}\rho((\zeta(\hat{x}-x)+\hat{x})-z),$$

where the last inequality follows from Lemma 2.

Since we have shown that $u(x,z) - u(x,(\zeta(\hat{x}-x)+\hat{x})) \geq \rho\hat{p}((\zeta(\hat{x}-x)+\hat{x})-z)$. This leads to a contradiction and establishes the desired result.

Subregion $(a_C, \hat{x}/(1-d))$**:**

Last, we consider $(a_C, \hat{x}/(1-d))$. Suppose on the contrary there exists $x \in (a_C, \hat{x}/(1-d))$ and $z \in h(x)$ such that $z \notin [(1-d)x, \hat{x}]$. Then we must have $z > \hat{x}$. By optimality of z,

$$V(x) = u(x,z) + \rho V(z) \geq u(x,\hat{x}) + \rho V(\hat{x}).$$

Rearranging the equation, we have

$$u(x,\hat{x}) - u(x,z) \leq \rho(V(z) - V(\hat{x})) \leq \rho V'_+(\hat{x})(z-\hat{x}) \leq \rho\hat{p}(z-\hat{x}),$$

where the second inequality follows from concavity of V, and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (a_C, \hat{x}/(1-d))$, we have

$$u(x,\hat{x}) - u(x,z) \geq (1/b)(z-\hat{x}) > \hat{p}\rho(z-\hat{x}),$$

where the last inequality follows from Lemma 2.

Since we have shown that $u(x,\hat{x}) - u(x,z) \leq \rho\hat{p}(z-\hat{x})$. This leads to a contradiction and establishes the desired result.

Proof of Corollary 1:

First, $h(\hat{x}) = \{\hat{x}\}$. If the initial stock is the golden rule stock, the system will stay at the golden rule stock. To see the dynamics for the initial stock $x \neq \hat{x}$, we rewrite $\zeta < 1$ more explicitly as

$$\frac{b}{a_C - a_I} - (1-d) < 1 \Leftrightarrow b - (1-d)(a_C - a_I) < a_C - a_I \Leftrightarrow [\theta a_I - a_C] + [a_I - (1-d)a_C] < 0,$$

where the last inequality suggests either (i) $\theta a_I < a_C$ or (ii) $a_I < (1-d)a_C$ (or both).

Consider (i) $\theta a_I < a_C$. Suppose $x \in [a_I, \hat{x})$. We know $G(x) = [\hat{x}, \zeta(\hat{x}-x)+\hat{x}]$. Since $\theta a_I < a_C$, $G(x) \subset [\hat{x}, \theta a_I] \subset [\hat{x}, a_C]$, and therefore, $G^2(x) = [(1-\zeta^2)\hat{x} + \zeta^2 x, \hat{x}]$. Since $\zeta \in (0,1)$ and $x < \hat{x}$, we have $(1-\zeta^2)\hat{x} + \zeta^2 x > x$. This implies that $\lim_{t\to\infty} G^{2t}(x) = \{\hat{x}\}$. Since $\lim_{x\to\hat{x}} \zeta(\hat{x}-x)+\hat{x} = \hat{x}$, we must have $\lim_{t\to\infty} G^{2t+1}(x) = \{\hat{x}\}$. This leads to the desired conclusion that x converges to $\hat{x}$ for $x \in [a_I, \hat{x})$. Since $\lim_{t\to\infty} G^t(x) = \{\hat{x}\}$ for any $x \in [a_I, \hat{x})$, $\lim_{t\to\infty} G^t(x) = \{\hat{x}\}$ for any $x \in G([a_I, \hat{x})) = [\hat{x}, \theta a_I]$. Further, since

$G([\hat{x}/\theta, a_I]) = [\hat{x}, \theta a_I]$, the system must converge for any x in $[\hat{x}/\theta, a_I]$. Since we know that $h(x) = \{\theta x\}$ for x in $(0, \hat{x}/\theta)$, for any x in $(0, \hat{x}/\theta)$, after finite periods, the stock must enter the region $[\hat{x}/\theta, \hat{x})$, thus leading to convergence. So far we have shown that the system converges for any x in $(0, \theta a_I]$. According to Theorem 1, for any x greater than $\hat{x}$, after finite periods, the stock must be below $\hat{x}$, again according to what we have shown, leading to convergence.

Consider (ii) $a_I < (1-d)a_C$. Suppose $x \in (\hat{x}, a_C]$. We know $G(x) = [\zeta(\hat{x}-x)+\hat{x}, \hat{x}]$. Since $a_I < (1-d)a_C$, $G(x) \subset [(1-d)a_C, \hat{x}] \subset [a_I, \hat{x}]$, and therefore, $G^2(x) = [\hat{x}, (1-\zeta^2)\hat{x} + \zeta^2 x]$. Since $\zeta \in (0,1)$ and $x > \hat{x}$, we have $(1-\zeta^2)\hat{x} + \zeta^2 x < x$. This implies that $\lim_{t\to\infty} G^{2t}(x) = \{\hat{x}\}$. Since $\lim_{x\to\hat{x}} \zeta(\hat{x}-x) + \hat{x} = \hat{x}$, we must have $\lim_{t\to\infty} G^{2t+1}(x) = \{\hat{x}\}$. This leads to the desired conclusion that x converges to $\hat{x}$ for $x \in (\hat{x}, a_C]$. Since $\lim_{t\to\infty} G^t(x) = \{\hat{x}\}$ for any $x \in (\hat{x}, a_C]$, $\lim_{t\to\infty} G^t(x) = \{\hat{x}\}$ for any $x \in G((\hat{x}, a_C]) = [(1-d)a_C, \hat{x}]$. Further, since $G([a_C, \hat{x}/(1-d))) = [(1-d)a_C, \hat{x}]$, the system must converge for any x in $[a_C, \hat{x}/(1-d))$. Since we know that $h(x) = \{(1-d)x\}$ for x in $[\hat{x}/(1-d), \infty)$, for any x in $[\hat{x}/(1-d), \infty)$, after finite periods, the stock must enter the region $([\hat{x}, \hat{x}/(1-d))$, thus leading to convergence. So far we have shown that the system converges for any x in $[(1-d)a_C, \infty)$. According to Theorem 1, for any x less than $\hat{x}$, after finite periods, the stock must be above $\hat{x}$, again according to what we have shown, leading to convergence.

We now have shown for any x, the optimal policy leads to a convergence to the golden rule stock.

Proof of Proposition 3:

Consider the subregion $(0, a_I]$.

Suppose on the contrary there exists $x \in (0, a_I]$ and $z \in h(x)$ such that $z \neq \theta x$. Then we must have $z < \theta x$. By the optimality of z,

$$V(x) = u(x,z) + \rho V(z) \geq u(x, \theta x) + \rho V(\theta x).$$

Rearranging the equation, we have

$$u(x,z) - u(x,\theta x) \geq \rho(V(\theta x) - V(z)) \geq \rho V'_-(\theta x)(\theta x - z) \geq \rho V'_-(\hat{x})(\theta x - z) \geq \rho\hat{p}(\theta x - z),$$

where the second inequality follows from concavity of V, the third inequality follows from concavity of V and the fact that $\theta a_I \leq \hat{x}$ for $\zeta \leq 0$ (Lemma 5), and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \leq a_I$, we have

$$u(x,z) - u(x,\theta x) = (1/a_C)[x - (a_I/b)(z - (1-d)x)] = \frac{a_I}{ba_C}(\theta x - z) < \hat{p}\rho(\theta x - z),$$

where the last inequality follows from Lemma 2. Since we have shown that $u(x,z) - u(x,\theta x) \geq \rho\hat{p}(\theta x - z)$. This leads to a contradiction and establishes the desired result.

Consider the subregion $[a_C, \infty)$.

Suppose on the contrary there exists $x \geq a_C$ and $z \in h(x)$ such that $z \neq (1-d)x$. Then we must have $z > (1-d)x$. By the optimality of z,

$$V(x) = u(x, z) + \rho V(z) \geq u(x, (1-d)x) + \rho V((1-d)x).$$

Rearranging the equation, we have

$$\begin{aligned} u(x,(1-d)x) - u(x,z) &\leq \rho(V(z) - V((1-d)x)) \leq \rho V'_+((1-d)x)(z-(1-d)x) \\ &\leq \rho V'_+(\hat{x})(z-(1-d)x) \leq \rho\hat{p}(z-(1-d)x), \end{aligned}$$

where the second inequality follows from concavity of V, the third inequality follows from concavity of V and the fact that $\hat{x} \leq (1-d)a_C \leq (1-d)x$ (Lemma 5), and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \geq a_C$, we have

$$u(x,(1-d)x) - u(x,z) = (z-(1-d)x)/b > \hat{p}\rho(z-(1-d)x),$$

where the last inequality follows from Lemma 2. Since we have shown that $u(x,(1-d)x) - u(x,z) \leq \rho\hat{p}(z-(1-d)x)$. This leads to a contradiction and establishes the desired result.

Consider the subregion $(a_I, \hat{x}]$.

Suppose on the contrary there exists $x \in (a_I, \hat{x}]$ and $z \in h(x)$ such that $z \notin [\zeta(\hat{x}-x)+\hat{x}, \hat{x}]$. There are two possible cases: (i) $z > \hat{x}$; (ii) $z < \zeta(\hat{x}-x)+\hat{x}$.

Consider (i) $z > \hat{x}$. We have

$$V(x) = u(x, z) + \rho V(z) \geq u(x, \hat{x}) + \rho V(\hat{x}).$$

Rearranging the equation, we have

$$u(x,\hat{x}) - u(x,z) \leq \rho(V(z) - V(\hat{x})) \leq \rho V'_+(\hat{x})(z-\hat{x}) \leq \rho\hat{p}(z-\hat{x}),$$

where the second inequality follows from concavity of V and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (a_I, \hat{x}]$ and $\zeta \leq 0$, we have

$$u(x,\hat{x}) - u(x,z) \geq (1/b)(z-\hat{x}) > \hat{p}\rho(z-\hat{x}),$$

where the last inequality follows from Lemma 2.

Since we have shown that $u(x,\hat{x}) - u(x,z) \leq \rho\hat{p}(z-\hat{x})$. This leads to a contradiction and establishes the desired result.

Consider (ii) $z < \zeta(\hat{x} - x) + \hat{x}$. We have

$$V(x) = u(x, z) + \rho V(z) \geq u(x, \zeta(\hat{x} - x) + \hat{x}) + \rho V(\zeta(\hat{x} - x) + \hat{x}).$$

Rearranging the equation, we have

$$\begin{aligned} u(x, z) - u(x, \zeta(\hat{x} - x) + \hat{x}) &\geq \rho(V(\zeta(\hat{x} - x) + \hat{x}) - V(z)) \\ &\geq \rho V'_{-}(\zeta(\hat{x} - x) + \hat{x})((\zeta(\hat{x} - x) + \hat{x}) - z) \\ &\geq \rho V'_{-}(\hat{x})((\zeta(\hat{x} - x) + \hat{x}) - z) \geq \rho\hat{p}((\zeta(\hat{x} - x) + \hat{x}) - z) \end{aligned}$$

where the second inequality follows from concavity of V, the third inequality follows from concavity of V and the fact that $\zeta(\hat{x} - x) + \hat{x} \leq \hat{x}$ for $x \in (a_I, \hat{x}]$, and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (a_I, \hat{x}]$, we have

$$u(x, z) - u(x, \zeta(\hat{x} - x) + \hat{x}) \leq \frac{a_I}{a_C b}((\zeta(\hat{x} - x) + \hat{x}) - z) < \hat{p}\rho((\zeta(\hat{x} - x) + \hat{x}) - z),$$

where the last inequality follows from Lemma 2.

Since we have shown that $u(x, z) - u(x, (\zeta(\hat{x} - x) + \hat{x})) \geq \rho\hat{p}((\zeta(\hat{x} - x) + \hat{x}) - z)$. This leads to a contradiction and establishes the desired result.

Last, consider the subregion $(\hat{x}, a_C)$.

Suppose on the contrary there exists $x \in (\hat{x}, a_C)$ and $z \in h(x)$ such that $z \notin [\hat{x}, \zeta(\hat{x} - x) + \hat{x}]$. There are two possible cases: (i) $z < \hat{x}$; (ii) $z > \zeta(\hat{x} - x) + \hat{x}$.

Consider (i) $z < \hat{x}$. We have

$$V(x) = u(x, z) + \rho V(z) \geq u(x, \hat{x}) + \rho V(\hat{x}).$$

Rearranging the equation, we have

$$u(x, z) - u(x, \hat{x}) \geq \rho(V(\hat{x}) - V(z)) \geq \rho V'_{-}(\hat{x})(\hat{x} - z) \geq \rho\hat{p}(\hat{x} - z),$$

where the second inequality follows from concavity of V and the third inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (\hat{x}, a_C)$ and $\zeta \leq 0$, we have

$$u(x, z) - u(x, \hat{x}) \leq \frac{a_I}{b a_C}(\hat{x} - z) < \hat{p}\rho(\hat{x} - z),$$

where the last inequality follows from Lemma 2.

Since we have shown that $u(x, z) - u(x, \hat{x}) \geq \rho\hat{p}(\hat{x} - z)$. This leads to a contradiction and establishes the desired result.

Consider (ii) $z > \zeta(\hat{x} - x) + \hat{x}$. We have

$$V(x) = u(x, z) + \rho V(z) \geq u(x, \zeta(\hat{x} - x) + \hat{x}) + \rho V(\zeta(\hat{x} - x) + \hat{x}).$$

Rearranging the equation, we have

$$\begin{aligned} u(x, \zeta(\hat{x} - x) + \hat{x}) - u(x, z) &\leq \rho(V(z) - V(\zeta(\hat{x} - x) + \hat{x})) \\ &\leq \rho V'_+(\zeta(\hat{x} - x) + \hat{x})(z - (\zeta(\hat{x} - x) + \hat{x})) \\ &\leq \rho V'_+(\hat{x})(z - (\zeta(\hat{x} - x) + \hat{x})) \leq \rho\hat{p}(z - (\zeta(\hat{x} - x) + \hat{x})) \end{aligned}$$

where the second inequality follows from concavity of V, the third inequality follows from concavity of V and the fact that $\zeta(\hat{x} - x) + \hat{x} \geq \hat{x}$ for $x \in (\hat{x}, a_C)$, and the last inequality follows from Lemma 4 ii).

By the definition of u, and given that $x \in (\hat{x}, a_C)$ we have

$$u(x, \zeta(\hat{x} - x) + \hat{x}) - u(x, z) \geq (1/b)(z - (\zeta(\hat{x} - x) + \hat{x})) > \hat{p}\rho(z - (\zeta(\hat{x} - x) + \hat{x})),$$

where the last inequality follows from Lemma 2.

Since we have shown that $u(x, (\zeta(\hat{x} - x) + \hat{x})) - u(x, z) \leq \rho\hat{p}(z - (\zeta(\hat{x} - x) + \hat{x}))$. This leads to a contradiction and establishes the desired result.

Proof of Proposition 4:

It has been covered in Proposition 3 for $x \in (0, a_I] \cup [a_C, \infty)$. We only need to consider $x \in (a_I, a_C)$. Suppose x is in $(a_I, \hat{x}]$. Since $\zeta > -1$ and $\zeta \leq 0$, we have $x < \zeta(\hat{x} - x) + \hat{x} \leq \hat{x}$, which implies $\zeta(\hat{x} - x) + \hat{x} \in (a_I, a_C)$. Similarly, if $x \in (\hat{x}, a_C)$, we must have $\zeta(\hat{x} - x) + \hat{x} \in (a_I, a_C)$. This suggests that for any $x \in (a_I, a_C)$, if we follow the policy such that $x' = \zeta(\hat{x} - x) + \hat{x}$, the stock of the next period is also in (a_I, a_C) and therefore, the policy that fully utilizes resources for $x \in (a_I, a_C)$ leads to zero total value loss. Any deviation from this policy leads to a positive value loss for $x \in (a_I, a_C)$, and therefore, it is not optimal. Hence, according to Lemma 8, $h(x) = \{\zeta(\hat{x} - x) + \hat{x}\}$ for $x \in (a_I, a_C)$.

Proof of Proposition 5:

Since ζ is in $(0, 1]$, or more explicitly, $b(a_C - a_I) - (1 - d) \leq 1$, rearranging the terms, we must have $(a_C - \theta a_I) + ((1 - d)a_C - a_I) \geq 0$, which implies that at least one of the following two inequalities holds: (A) $a_C \geq \theta a_I$; (B) $(1 - d)a_C \geq a_I$. Therefore, we consider three possible cases.

(i) Both (A) and (B) hold: $a_C \geq \theta a_I$ and $(1 - d)a_C \geq a_I$.

This is the simplest case. Consider $x \in [a_I, a_C]$. Since $f(x) \equiv \zeta(\hat{x} - x) - \hat{x}) \in [a_C(1-d), a_I\theta] \subset [a_I, a_C]$, the sequence of the capital stock generated by f, $\{f^t(x)\}_{t=1}^{\infty}$, is bounded by $[a_I, a_C]$. Further, since we know from Lemma 1 that the value loss associated with $(x, f(x))$ is zero for $x \in [a_I, a_C]$, the sum of the discounted value losses associated with $\{f^t(x)\}_{t=1}^{\infty}$ is zero. Stating from x, any program that deviates from $\{f^t(x)\}_{t=1}^{\infty}$ yields a positive value loss. According to Lemma 8, $h(x) = \{\zeta(\hat{x} - x) - \hat{x})\}$ for $x \in [a_I, a_C]$.

Now consider $x \in (\hat{x}/\theta, a_I)$. According to Theorem 1, we know $h(x) \subset [\hat{x}, \theta x] \subset [\hat{x}, \theta a_I] \subset [\hat{x}, a_C]$. Since we know that the total value loss for the optimal program starting from $x \in [a_I, a_C]$ is always zero, we just need to check the one-period value loss for (x, x') with $x \in (\hat{x}/\theta, a_I)$ and $x' \in [\hat{x}, \theta x]$:

$$\begin{aligned}\delta^\rho(x, x') &= u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} - u(x, x') - \hat{p}(\rho x' - x) \\ &= u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} - (1/a_C)(x - (a_I/b)(x' - (1 - d)x)) - \hat{p}(\rho x' - x).\end{aligned}$$

Then we have

$$\frac{\partial \delta^\rho(x, x')}{\partial x'} = \frac{a_I}{a_C b} - \hat{p}\rho < 0,$$

where the inequality follows from Lemma 2. Since the one-period value loss strictly decreases with x', it attains its unique minimum and therefore the total value loss attains its unique minimum, when x' attains its unique maximum, which implies that $h(x) = \{\theta x\}$ for $x \in (\hat{x}/\theta, a_I)$.

Consider $x \in (a_C, \hat{x}/(1 - d))$. According to Theorem 1, we know $h(x) \subset [(1 - d)x, \hat{x}] \subset [(1 - d)a_C, \hat{x}] \subset [a_I, \hat{x}]$. Since we know that the total value loss for the optimal program starting from $x \in [a_I, a_C]$ is always zero, we just need to check the one-period value loss for (x, x') with $x \in (a_C, \hat{x}/(1 - d))$ and $x' \in [(1 - d)x, \hat{x}]$:

$$\delta^\rho(x, x') = u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} - (1 - (1/b)(x' - (1 - d)x)) - \hat{p}(\rho x' - x).$$

Then we have

$$\frac{\partial \delta^\rho(x, x')}{\partial x'} = \frac{1}{b} - \hat{p}\rho > 0,$$

where the inequality follows from Lemma 2. Since the one-period value loss strictly increases with x', it attains its unique minimum and therefore the total value loss attains its unique minimum, when x' attains its unique minimum, which implies $h(x) = \{(1 - d)x\}$ for $x \in (a_C, \hat{x}/(1 - d))$.

Combined with the characterization for $x \in (0, \hat{x}/\theta] \cup [\hat{x}/(1-d), \infty)$ as in Theorem 1, we have obtained the desired result for case (i).

(ii) Only (B) holds: $a_C < \theta a_I$ and $(1 - d)a_C \geq a_I$.

The complication arises from the fact that $a_C < a_I\theta$. As $a_C < a_I\theta$, $f(a_I) = \zeta(\hat{x} - a_I) + \hat{x} = a_I\theta > a_C$, which means, $f(a_I) \notin [a_I, a_C]$. The total value loss could be strictly positive even if we follow the policy f with an initial stock starting from a_I.

Consider $x \in [\hat{x}, a_C]$. Since $a_C(1 - d) \geq a_I$, $f(x) = \zeta(\hat{x} - x) + \hat{x} \in [(1 - d)a_C, \hat{x}] \subset [a_I, \hat{x}]$. Since $f(x) \in [a_I, \hat{x}]$, $f^2(x) = \zeta^2 x + (1 - \zeta^2)\hat{x} \in [\hat{x}, x] \subset [\hat{x}, a_C]$, where $\zeta^2 x + (1 -$

$\zeta^2)\hat{x} \le x$ follows from $\zeta \in (0,1]$ and $x \ge \hat{x}$. Therefore, $\{f^t(x)\}_{t=1}^{\infty}$, is bounded by $[a_I, a_C]$. It follows from the argument for case (i) that $h(x) = \{\zeta(\hat{x} - x) + \hat{x}\}$ for $x \in [\hat{x}, a_C]$.

Consider $x \in [a_C(\zeta - d)/\zeta, \hat{x})$. Since $a_C < a_I\theta$, $a_C(\zeta - d)/\zeta > a_I$. Since $f(x) \in (\hat{x}, a_C]$ with $\delta^\rho(x, f(x)) = 0$ and we have shown that the optimal policy function leads to the total value loss being zero for any initial stock in $(\hat{x}, a_C]$, we must have $h(x) = \{\zeta(\hat{x} - x) + \hat{x}\}$ for $x \in [a_C(\zeta - d)/\zeta, \hat{x})$.

Consider $x \in (a_C, \hat{x}/(1-d))$. According to Theorem 1, we know $h(x) \subset [(1-d)x, \hat{x}] \subset [(1-d)a_C, \hat{x}] \subset [a_C(\zeta - d)/\zeta, \hat{x}]$, where $[(1-d)a_C, \hat{x}] \subset [a_C(\zeta-d)/\zeta, \hat{x}]$ follows from $\zeta \in (0,1]$. Then it follows from the argument for case (i) that $h(x) = \{(1-d)x\}$ for $x \in (a_C, \hat{x}/(1-d))$.

Consider $x \in (\hat{x}/\theta, a_C/\theta]$. Since $a_C < a_I\theta$, $a_C/\theta < a_I$. According to Theorem 1, we know $h(x) \subset [\hat{x}, \theta x] \subset [\hat{x}, a_C]$. Again, it follows from the argument for case (i) that $h(x) = \{\theta x\}$ for $x \in (\hat{x}/\theta, a_C/\theta]$.

Last, consider $x \in (a_C/\theta, a_C(\zeta-d)/\zeta)$.According to Theorem 1, $h(x) \subset [\hat{x}, \min\{\theta x, \zeta(\hat{x} - x) + \hat{x}\}]$. Since x is in $(a_C/\theta, a_C(\zeta-d)/\zeta)$, we have $[\hat{x}, a_C] \subset [\hat{x}, \min\{\theta x, \zeta(\hat{x} - x) + \hat{x}\}]$. Let $x' \in [\hat{x}, \min\{\theta x, \zeta(\hat{x} - x) + \hat{x}\}]$. If $x' \le a_C$, then the total value loss is simply the one period value loss $\delta^\rho(x, x')$. Following the argument for case (i), the one period value loss is minimized when x' attains its maximum, a_C. Hence, we must have $h(x) \subset [a_C, \min\{\theta x, \zeta(\hat{x} - x) + \hat{x}\}]$.

Combined with the characterization for $x \in (0, \hat{x}/\theta] \cup [\hat{x}/(1-d), \infty)$ in Theorem 1, we have obtained the desired result for case (ii).

(iii) Only (A) holds: $a_C \ge \theta a_I$ and $(1-d)a_C < a_I$.

The complication for this case arises from the fact that $a_C(1-d) < a_I$. As $a_C(1-d) < a_I$, $f(a_C) = \zeta(\hat{x} - a_C) + \hat{x} = (1-d)a_C < a_I$, which means $f(a_C) \notin [a_I, a_C]$. The total value loss could be strictly positive even if we follow the policy f with an initial stock starting from a_C.

Consider $x \in [a_I, \hat{x}]$. Since $a_C \ge a_I\theta$, it follows symmetrically from the argument for $[\hat{x}, a_C]$ in case (ii) that $h(x) = \{\zeta(\hat{x} - x) + \hat{x}\}$ for $x \in [a_I, \hat{x}]$.

Consider $x \in (\hat{x}, a_C(1+(1-d)/\zeta) - a_I/\zeta]$. Since $a_C(1-d) < a_I$, $a_C(1+(1-d)/\zeta) - a_I/\zeta < a_C$. Then it follows symmetrically from the argument for $[a_C(\zeta-d)/\zeta, \hat{x})$ in case (ii) that $h(x) = \{\zeta(\hat{x} - x) + \hat{x}\}$ for $x \in (\hat{x}, a_C(1 + (1-d)/\zeta) - a_I/\zeta]$.

Consider $x \in (\hat{x}/\theta, a_I)$. According to Theorem 1, $h(x) \subset [\hat{x}, \theta x] \subset [\hat{x}, \theta a_I] \subset [\hat{x}, a_C(1+(1-d)/\zeta) - a_I/\zeta]$, where the last $\subset$ holds because $\theta a_I \le a_C(1+(1-d)/\zeta) - a_I/\zeta$, which itself follows from $f(\theta a_I) \ge a_I$ (due to $\zeta \le 1$), $f(a_C(1 + (1-d)/\zeta) - a_I/\zeta) = a_I$, and f being decreasing. Then it follows from the argument for case (i) that $h(x) = \{\theta x\}$ for $x \in (\hat{x}/\theta, a_I)$.

Consider $x \in (a_I/(1-d), \hat{x}/(1-d))$. Since $a_C(1-d) < a_I$, $a_I/(1-d) > a_C$. According to Theorem 1, $h(x) \subset [(1-d)x, \hat{x}] \subset [a_I, \hat{x}]$. It then follows from the argument for case (i) that $h(x) = \{(1-d)x\}$ for $x \in (a_I/(1-d), \hat{x}/(1-d))$.

Last, consider $x \in (a_C(1+(1-d)/\zeta) - a_I/\zeta, a_I/(1-d))$. According to Theorem 1, $h(x) \subset [\max\{(1-d)x, \zeta(\hat{x}-x)+\hat{x}\}, \hat{x}]$. Since x is in $(a_C(1+(1-d)/\zeta)-a_I/\zeta, a_I/(1-d))$, we have $[a_I, \hat{x}] \subset [\max\{(1-d)x, \zeta(\hat{x}-x)+\hat{x}\}, \hat{x}]$. If $x' \geq a_I$, then total value loss is simply the one period value loss. Following the argument for case (i), the one period value loss is minimized when x' attains its minimum, a_I. Then we must have $h(x) \subset [\max\{(1-d)x, \zeta(\hat{x}-x)+\hat{x}\}, a_I]$.

Combined with the characterization for $x \in (0, \hat{x}/\theta] \cup [\hat{x}/(1-d), \infty)$ in Theorem 1, we have obtained the desired result for case (iii).

Proof of Theorem 2:

We first show the first part of the proposition concerning the definition and the order of $\bar{\rho}_t$. Let $f_t(\rho) \equiv a_C b(1-d)^t \rho^{t+1} - (a_C - a_I)a_I\zeta\rho - a_I(a_C - a_I)$. Since $f_t(0) < 0$ and $f_t(\rho) > 0$ for ρ sufficiently large, there must exist at least one positive root to the equation $f_t(\rho) = 0$. Suppose there are two different roots, denoted by ρ_1 and ρ_2. Without loss of generality, let $\rho_1 > \rho_2$. Then we have

$$\begin{aligned} a_C b(1-d)^t \rho_1^{t+1} - (a_C - a_I)a_I\zeta\rho_1 - a_I(a_C - a_I) &= 0 \\ a_C b(1-d)^t \rho_2^{t+1} - (a_C - a_I)a_I\zeta\rho_2 - a_I(a_C - a_I) &= 0, \end{aligned}$$

which implies

$$\begin{aligned} & a_C b(1-d)^t(\rho_1^{t+1} - \rho_2^{t+1}) = (a_C - a_I)a_I\zeta(\rho_1 - \rho_2) \\ \Leftrightarrow \quad & (a_C - a_I)a_I\zeta = \frac{a_C b(1-d)^t(\rho_1^{t+1} - \rho_2^{t+1})}{\rho_1 - \rho_2} > a_C b(1-d)^t \rho_2^t, \end{aligned}$$

where the last equality follows from $\rho_1 > \rho_2$. Since $(a_C - a_I)a_I\zeta > a_C b(1-d)^t\rho_2^t$, $a_C b(1-d)^t\rho_2^{t+1} - (a_C - a_I)a_I\zeta\rho_2 - a_I(a_C - a_I) < 0$, leading to the contradiction. Hence, $\bar{\rho}_t$ is the unique positive root, being well-defined. Further, since $f_1(1/\theta) = ba_C(1-d-\theta)/\theta^2 < 0$ and we know $f_1(\rho)$ is positive for ρ sufficiently large, $\bar{\rho}_1 > 1/\theta$. Since $f_t(1/(1-d)) = b(a_C - a_I)/(1-d) > 0$ and we know $f_t(0) < 0$, $\bar{\rho}_t < 1/(1-d)$ for any t.

By definition, we have

$$\begin{aligned} f_{t+1}(\bar{\rho}_{t+1}) = 0 \Leftrightarrow a_C b(1-d)^{t+1}\bar{\rho}_{t+1}^{t+2} - (a_C - a_I)a_I\zeta\bar{\rho}_{t+1} - a_I(a_C - a_I) = 0 \\ f_t(\bar{\rho}_t) = 0 \Leftrightarrow a_C b(1-d)^t\bar{\rho}_t^{t+1} - (a_C - a_I)a_I\zeta\bar{\rho}_t - a_I(a_C - a_I) = 0. \end{aligned}$$

Since $\bar{\rho}_{t+1} < 1/(1-d)$, or equivalently, $\bar{\rho}_{t+1}(1-d) < 1$,

$$f_{t+1}(\bar{\rho}_t) = a_C b(1-d)^{t+1}\bar{\rho}_t^{t+2} - (a_C - a_I)a_I\zeta\bar{\rho}_t - (a_C - a_I)a_I < f_t(\bar{\rho}_t) = 0.$$

Further, we know $f_t(\rho) > 0$ for ρ sufficiently large, so $\bar{\rho}_{t+1} > \bar{\rho}_t$.

Now we turn to characterizing the optimal policy correspondence for $x \in (a_C/\theta, a_C(\zeta - d)/\zeta)$.

Pick the smallest integer t_0 such that $a_I\theta(1-d)^{t_0} < a_C$. By construction, $t_0 \geq 1$ and $a_I\theta(1-d)^{t_0-1} \geq a_C$, so $a_I\theta(1-d)^{t_0} \geq (1-d)a_C \geq a_C(\zeta - d)/\zeta$, where the last inequality follows from $0 < \zeta \leq 1$.

Pick $x \in (a_C/\theta, a_C(\zeta - d)/\zeta)$. According to case (ii) in Proposition 5, the stock for the next period, x', has to be in $[a_C, \min\{\zeta(\hat{x}-x)+\hat{x}, \theta x\}]$, so $x' \leq a_I\theta$. Pick the smallest integer t_1 such that $(1-d)^{t_1}x' < a_C$. Since $x' \leq a_I\theta$, by construction, $1 \leq t_1 \leq t_0$ and $(1-d)^{t_1-1}x' \geq a_C$, so $(1-d)^{t_1}x' \geq (1-d)a_C \geq a_C(\zeta-d)/\zeta$.

For any stock above a_C, notice that the optimality mandates the stock in the following period to shirk by $(1-d)$ times. Following x', the stock for the next t_1 periods are given by $\{(1-d)^t x'\}_{t=1}^{t_1}$. Since $(1-d)^{t_1}x' \subset [a_C(\zeta-d)/\zeta, a_C)$, after t_1+1 periods, the total value loss of the remaining periods will be zero, so we focus on the total value loss for the first t_1+1 periods.

Consider the (t_1+1)-period value loss associated with (x, x') and $\{((1-d)^t x', (1-d)^{t+1}x')\}_{t=0}^{t=t_1-1}$.

$$\begin{aligned}
\ell_{t_1}(x') &\equiv \delta^\rho(x,x') + \sum_{t=0}^{t_1-1} \rho^{t+1}\delta^\rho((1-d)^t x', (1-d)^{t+1}x') \\
&= \frac{1-\rho^{t_1}}{1-\rho}u(\hat{x},\hat{x}) + (\rho^{t_1}-1)\hat{p}\hat{x} - (1/a_C)(x - (a_I/b)(x' - (1-d)x)) \\
&\quad -\frac{\rho - \rho^{t_1+1}}{1-\rho} - \hat{p}(\rho^{t_1+1}(1-d)^{t_1}x' - x)
\end{aligned}$$

Then we have

$$\frac{\partial \ell_{t_1}(x')}{\partial x'} = \frac{a_I}{ba_C} - \hat{p}\rho^{t_1+1}(1-d)^{t_1} = \frac{a_I}{ba_C} - \frac{\rho^{t_1+1}(1-d)^{t_1}}{(a_C - a_I)(1+\rho\zeta)} = \frac{-f_{t_1}(\rho)}{ba_C(a_C-a_I)(1+\rho\zeta)}.$$

By construction of $\bar{\rho}_{t_1}$, we know $\partial\ell_{t_1}(x')/\partial x' > 0$ if $\rho < \bar{\rho}_{t_1}$; $\partial\ell_{t_1}(x')/\partial x' = 0$ if $\rho = \bar{\rho}_{t_1}$; $\partial\ell_{t_1}(x')/\partial x' < 0$ if $\rho > \bar{\rho}_{t_1}$.

Consider two possible cases: (1) $t_0 = 1$; (2) $t_0 > 1$.

For (1), $t_0 = 1$, so we must have $t_1 = 1$. Hence, we only need to consider the two-period value loss. If $\rho > \bar{\rho}_1$, the total value loss attains its minimum when x' attains its maximum, suggesting that $h(x) = \min\{\zeta(\hat{x}-x)+\hat{x}, \theta x\}$. If $\rho = \bar{\rho}_1$, then the total value loss is constant with respect to x', so $h(x) = [a_C, \min\{\zeta(\hat{x}-x)+\hat{x}, \theta x\}]$. If $\rho < \bar{\rho}_1$, the total value loss attains its minimum when x' attains its minimum, which implies that $h(x) = \{a_C\}$.

For (2), $(1/\theta, \infty)$ is partitioned by $\{\bar{\rho}_t\}_{t=1}^{t_0}$: $(1/\theta, \bar{\rho}_1)$, $\{\bar{\rho}_1\}$, $(\bar{\rho}_1, \bar{\rho}_2)$,..., $(\bar{\rho}_{t_0-1}, \bar{\rho}_{t_0})$, $\{\bar{\rho}_{t_0}\}$, and $(\bar{\rho}_{t_0}, \infty)$.

Consider $\rho < \bar{\rho}_1$. The two-period value loss is minimized and total value loss is equal to the two-period value loss when $x' = a_C$, so $h(x) = \{a_C\}$.

Consider $\rho = \bar{\rho}_{t_1}$ for t_1 taking value from $\{1, 2, ..., t_0\}$. The $(t_1 + 1)$-period value loss and also the total value loss is constant with respect to x' for a fixed t_1. Since $\rho = \bar{\rho}_{t_1}$, $\rho > \bar{\rho}_{t_1'}$ for any $t_1' < t_1$ and $\rho < \bar{\rho}_{t_1'}$ for any $t_1' > t_1$. Since $\rho > \bar{\rho}_{t_1'}$ for any $t_1' < t_1$, the total value loss decreases with x' for $t_1' < t_1$, or equivalently, for $x'(1-d)^{t_1-1} < a_C$.[5] Since $\rho < \bar{\rho}_{t_1'}$ for any $t_1' > t_1$, the total value loss increases with x' for $t_1' > t_1$, or equivalently, for $x'(1-d)^{t_1} \geq a_C$. If $\min\{\zeta(\hat{x}-x)+\hat{x}, \theta x\} > a_C/(1-d)^{t_1}$, then $h(x) = [a_C/(1-d)^{t_1-1}, a_C/(1-d)^{t_1}]$. If $\min\{\zeta(\hat{x}-x)+\hat{x}, \theta x\} \in [a_C/(1-d)^{t_1-1}, a_C/(1-d)^{t_1}]$, then $h(x) = [a_C/(1-d)^{t_1-1}, \min\{\zeta(\hat{x}-x)+\hat{x}, \theta x\}]$. If $\min\{\zeta(\hat{x}-x)+\hat{x}, \theta x\} < a_C/(1-d)^{t_1}$, then $h(x) = \min\{\zeta(\hat{x}-x)+\hat{x}, \theta x\}$. In sum, for $\rho = \bar{\rho}_{t_1}$, $h(x) = [\min\{\zeta(\hat{x}-x)+\hat{x}, \theta x, a_C/(1-d)^{t-1}\}, \min\{\zeta(\hat{x}-x)+\hat{x}, \theta x, a_C/(1-d)^{t}\}]$.

Consider $\rho \in (\bar{\rho}_{t_1}, \bar{\rho}_{t_1+1})$ for t_1 taking value from $\{1, ..., t_0 - 1\}$. The $(t_1 + 1)$-period value loss and also the total value loss is minimized when x' attains its maximum for a fixed t_1. Since $\rho < \bar{\rho}_{t_1+1}$, $\rho < \bar{\rho}_{t_1'}$ for any $t_1' > t_1$, which implies that the total value loss increases with x' for $t_1' > t_1$, or equivalently, for $x'(1-d)^{t_1} \geq a_C$. Since $\rho > \bar{\rho}_{t_1}$, $\rho > \bar{\rho}_{t_1'}$ for any $t_1' < t_1$, which implies that the total value loss decreases with x' for $t_1' < t_1$, or equivalently, for $x'(1-d)^{t_1-1} < a_C$. Hence, we have $h(x) = \min\{\zeta(\hat{x}-x)+\hat{x}, \theta x, a_C/(1-d)^{t_1}\}$.

Last, consider $\rho > \bar{\rho}_{t_0}$. Since we know $\bar{\rho}_{t_0} \geq \bar{\rho}_t$ for any $t = 1, 2, ..., t_0$, $\rho > \bar{\rho}_{t_1}$ for any t_1. This suggests that the $(t_1 + 1)$-period value loss and also the total value loss decreases with x' for *any* given t_1. Then the total value loss is minimized when x' attains its maximum. Hence, $h(x) = \min\{\zeta(\hat{x}-x)+\hat{x}, \theta x\}$.

We have now obtained the desired conclusion.

Proof of Theorem 3:

We first show the first part of the proposition concerning the definition and the order of $\tilde{\rho}_t$. Let $f_t(\rho) \equiv b\theta^t\rho^{t+1} - (a_C - a_I)\zeta\rho - (a_C - a_I)$. Since $f_t(0) < 0$ and $f_t(\rho) > 0$ for ρ sufficiently large, there must exist at least one positive root to the equation $f_t(\rho) = 0$. Suppose there are two different roots, denoted by ρ_1 and ρ_2. Without loss of generality, let $\rho_1 > \rho_2$. Then we have

$$\begin{aligned} b\theta^t\rho_1^{t+1} - (a_C - a_I)\zeta\rho_1 - (a_C - a_I) &= 0 \\ b\theta^t\rho_2^{t+1} - (a_C - a_I)\zeta\rho_2 - (a_C - a_I) &= 0, \end{aligned}$$

[5]Here we implicitly rely on the continuity of the value function.

which implies

$$b\theta^t(\rho_1^{t+1}-\rho_2^{t+1}) = (a_C-a_I)\zeta(\rho_1-\rho_2) \Leftrightarrow (a_C-a_I)\zeta = \frac{b\theta^t(\rho_1^{t+1}-\rho_2^{t+1})}{\rho_1-\rho_2} > b\theta^t\rho_2^t,$$

where the last equality follows from $\rho_1 > \rho_2$. Since $(a_C-a_I)\zeta > b\theta^t\rho_2^t$, $b\theta^t\rho_2^{t+1} - (a_C - a_I)\zeta\rho_2 - (a_C-a_I) < 0$, leading to the contradiction. Hence, $\tilde{\rho}_t$ is the unique positive root, being well-defined. Further, since $f_t(1/\theta) = -b(a_C-a_I)/(a_I\theta) < 0$ and we know $f_t(\rho)$ is positive for ρ sufficiently large, $\tilde{\rho}_t > 1/\theta$.

By definition, we have

$$\begin{aligned} f_{t+1}(\tilde{\rho}_{t+1}) = 0 \Leftrightarrow b\theta^{t+1}\tilde{\rho}_{t+1}^{t+2} - (a_C-a_I)\zeta\tilde{\rho}_{t+1} - (a_C-a_I) = 0 \\ f_t(\tilde{\rho}_t) = 0 \Leftrightarrow b\theta^t\tilde{\rho}_t^{t+1} - (a_C-a_I)\zeta\tilde{\rho}_t - (a_C-a_I) = 0. \end{aligned}$$

Since $\tilde{\rho}_{t+1} > 1/\theta$, or equivalently, $\tilde{\rho}_{t+1}\theta > 1$,

$$f_t(\tilde{\rho}_{t+1}) = b\theta^t\tilde{\rho}_{t+1}^{t+1} - (a_C-a_I)\zeta\tilde{\rho}_{t+1} - (a_C-a_I) < f_{t+1}(\tilde{\rho}_{t+1}) = 0.$$

Further, we know $f_t(\rho) > 0$ for ρ sufficiently large, so $\tilde{\rho}_{t+1} < \tilde{\rho}_t$.

Now we turn to characterizing the optimal policy correspondence for $x \in (a_C(1 + (1-d)/\zeta) - a_I/\zeta, a_I/(1-d))$.

Pick the smallest integer t_0 such that $\theta^{t_0}a_C(1-d) > a_I$. By construction, $t_0 \geq 1$ and $\theta^{t_0-1}a_C(1-d) \leq a_I$, so $\theta^{t_0}a_C(1-d) \leq \theta a_I \leq a_C(1+(1-d)/\zeta) - a_I/\zeta$, where the last inequality follows from $\zeta \leq 1$ and $a_I\theta \leq a_C$ (also see the proof for Proposition 5).

Pick $x \in (a_C(1+(1-d)/\zeta) - a_I/\zeta, a_I/(1-d))$. According to case (iii) in Proposition 5, the stock for the next period, x', has to be in $[\max\{\zeta(\hat{x}-x)+\hat{x}, (1-d)x\}, a_I]$, so $x' \geq a_C(1-d)$. Pick the smallest integer t_1 such that $\theta^{t_1}x' > a_I$. Since $x' \geq a_C(1-d)$, by construction, $1 \leq t_1 \leq t_0$ and $\theta^{t_1-1}a_C(1-d) \leq a_I$, so $\theta^{t_1}a_C(1-d) \leq \theta a_I \leq a_C(1 + (1-d)/\zeta) - a_I/\zeta$. For any stock below a_I, notice that the optimality mandates the stock in the following period to grow up by θ times. Following x', the stock for the next t_1 periods are given by $\{\theta^t x'\}_{t=1}^{t_1}$. Since $\theta^{t_1}x' \subset (a_I, a_C(1+(1-d)/\zeta) - a_I/\zeta]$, after t_1+1 periods, the total value loss of the remaining periods will be zero, so we focus on the total value loss for the first t_1+1 periods.

Consider the (t_1+1)-period value loss associated with (x, x') and $\{(\theta^t x', \theta^{t+1}x')\}_{t=0}^{t=t_1-1}$.

$$\begin{aligned} \ell_{t_1}(x') &\equiv \delta^\rho(x,x') + \sum_{t=0}^{t_1-1}\rho^{t+1}\delta^\rho(\theta^t x', \theta^{t+1}x') \\ &= \frac{1-\rho^{t_1}}{1-\rho}u(\hat{x},\hat{x}) + (\rho^{t_1}-1)\hat{p}\hat{x} - (1-(1/b)(x'-(1-d)x)) - \hat{p}(\rho^{t_1+1}\theta^{t_1}x' - x) \end{aligned}$$

Then we have

$$\frac{\partial \ell_{t_1}(x')}{\partial x'} = \frac{1}{b} - \hat{p}\rho^{t_1+1}\theta^{t_1} = \frac{1}{b} - \frac{\rho^{t_1+1}\theta^{t_1}}{(a_C - a_I)(1+\rho\zeta)} = \frac{-f_{t_1}(\rho)}{b(a_C - a_I)(1+\rho\zeta)}.$$

By construction of $\tilde{\rho}_{t_1}$, we know $\partial \ell_{t_1}(x')/\partial x' > 0$ if $\rho < \tilde{\rho}_{t_1}$; $\partial \ell_{t_1}(x')/\partial x' = 0$ if $\rho = \tilde{\rho}_{t_1}$; $\partial \ell_{t_1}(x')/\partial x' < 0$ if $\rho > \tilde{\rho}_{t_1}$.

Consider two possible cases: (1) $t_0 = 1$; (2) $t_0 > 1$.

For (1), $t_0 = 1$, so $t_1 = 1$. Hence, we only need to consider the two-period value loss. If $\rho < \tilde{\rho}_1$, the total value loss attains its minimum when x' attains its minimum, suggesting that $h(x) = \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\}$. If $\rho = \tilde{\rho}_1$, then the total value loss is constant with respect to x', so $h(x) = [\max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\}, a_I]$. If $\rho > \tilde{\rho}_1$, the total value loss attains its minimum when x' attains its maximum, which implies that $h(x) = \{a_I\}$.

For (2), $(1/\theta, \infty)$ is partitioned by $\{\tilde{\rho}_t\}_{t=1}^{t_0} : (1/\theta, \tilde{\rho}_{t_0})$, $\{\tilde{\rho}_{t_0}\}$, $(\tilde{\rho}_{t_0}, \tilde{\rho}_{t_0-1})$,..., $(\tilde{\rho}_2, \tilde{\rho}_1)$, $\{\tilde{\rho}_1\}$, and $(\tilde{\rho}_1, \infty)$.

Consider $\rho > \tilde{\rho}_1$. The two-period value loss is minimized and total value loss is equal to the two-period value loss when $x' = a_I$, so $h(x) = \{a_I\}$.

Consider $\rho = \tilde{\rho}_{t_1}$ for t_1 taking value from $\{1, 2, ..., t_0\}$. The $(t_1 + 1)$-period value loss and also the total value loss is constant with respect to x' for a fixed t_1. Since $\rho = \tilde{\rho}_{t_1}$, $\rho > \tilde{\rho}_{t_1'}$ for any $t_1' > t_1$ and $\rho < \tilde{\rho}_{t_1'}$ for any $t_1' < t_1$. Since $\rho > \tilde{\rho}_{t_1'}$ for any $t_1' > t_1$, the total value loss decreases with x' for $t_1' > t_1$, or equivalently, for $x'\theta^{t_1} \leq a_I$.[6] Since $\rho < \tilde{\rho}_{t_1'}$ for any $t_1' < t_1$, the total value loss increases with x' for $t_1' < t_1$, or equivalently, for $x'\theta^{t_1-1} > a_I$. If $\max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\} < a_I/\theta^{t_1}$, then $h(x) = [a_I/\theta^{t_1}, a_I/\theta^{t_1-1}]$. If $\max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\} \in [a_I/\theta^{t_1}, a_I/\theta^{t_1-1}]$, then $h(x) = [\max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\}, a_I/\theta^{t_1-1}]$. If $\max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\} > a_I/\theta^{t_1-1}$, then $h(x) = \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\}$. In sum, for $\rho = \tilde{\rho}_{t_1}$, $h(x) = [\max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x, a_I/\theta^{t_1}\}, \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x, a_I/\theta^{t_1-1}\}]$.

Consider $\rho \in (\tilde{\rho}_{t_1+1}, \tilde{\rho}_{t_1})$ for t_1 taking value from $\{1, ..., t_0 - 1\}$. The $(t_1 + 1)$-period value loss and also the total value loss is minimized when x' attains its minimum for a fixed t_1. Since $\rho > \tilde{\rho}_{t_1+1}$, $\rho > \tilde{\rho}_{t_1'}$ for any $t_1' > t_1$, which implies that the total value loss decreases with x' for $t_1' > t_1$, or equivalently, for $x'\theta^{t_1} \leq a_I$. Since $\rho < \tilde{\rho}_{t_1}$, $\rho < \tilde{\rho}_{t_1'}$ for any $t_1' < t_1$, which implies that the total value loss increases with x' for $t_1' < t_1$, or equivalently, for $x'\theta^{t_1-1} > a_I$. Hence, we have $h(x) = \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x, a_I/\theta^{t_1}\}$.

Last, consider $\rho < \tilde{\rho}_{t_0}$. Since we know $\tilde{\rho}_{t_0} \leq \tilde{\rho}_t$ for any $t = 1, 2, ..., t_0$, $\rho < \tilde{\rho}_{t_1}$ for any t_1. This suggests that the $(t_1 + 1)$-period value loss and also the total value loss increases with x' for *any* given t_1. Then the total value loss is minimized when x' attains its minimum. Hence, $h(x) = \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\}$.

[6] Here we implicitly rely on the continuity of the value function.

We have now obtained the desired conclusion.

2 Additional Illustration

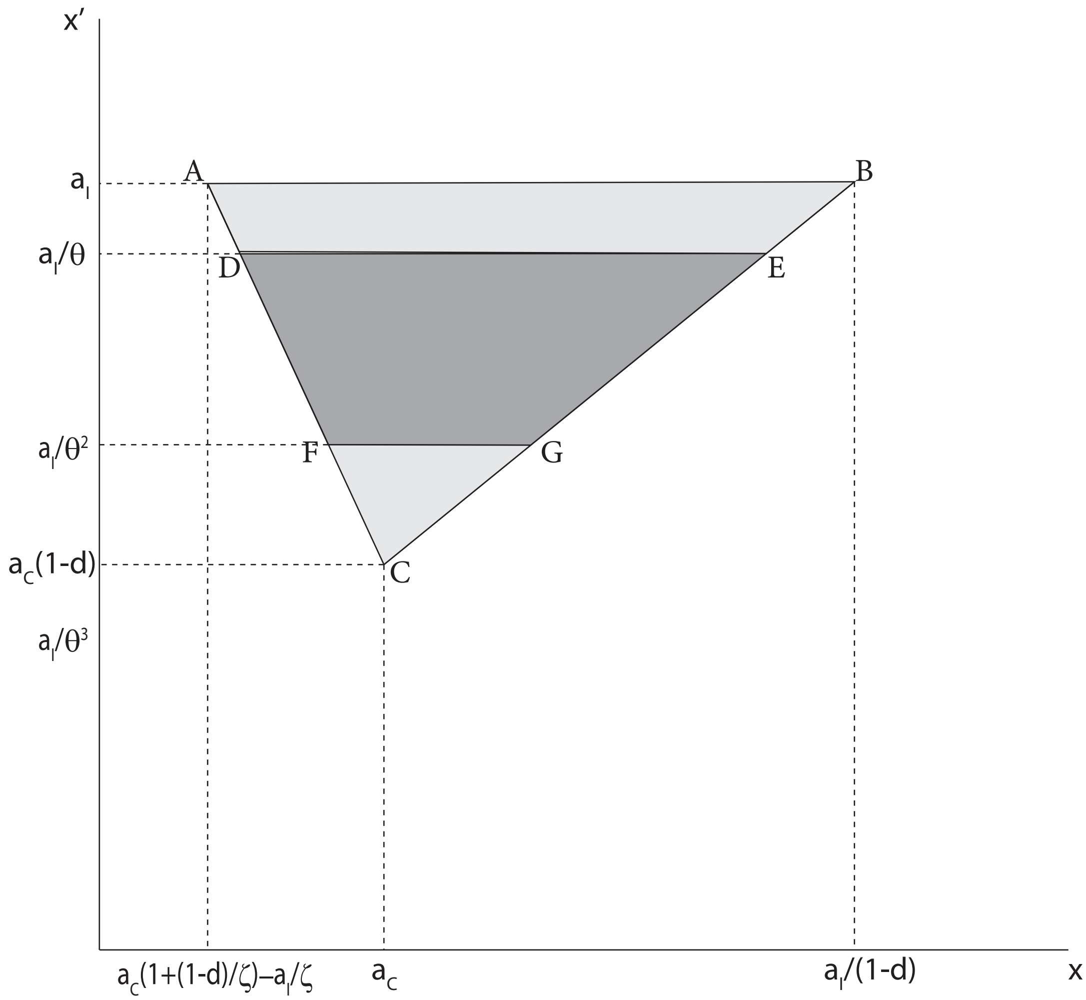

Figure 1: Illustration of Theorem 3

References

[1] Khan, M. Ali and Mitra, T., 2007, Optimal growth under discounting in the two-sector Robinson-Solow-Srinivasan model: a dynamic programming approach, *Journal of Difference Equations and Applications* **13**, 151-168.