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Optimal growth in the Robinson-Shinkai-Leontief model: the case of capital-intensive consumption goods

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Abstract:
We study the two-sector Robinson-Shinkai-Leontief (RSL) model of discrete-time optimal economic growth for the case of capital-intensive consumption goods. We frame the model in the context of Nishimura's *oeuvre,* and more specifically, relate it to its neoclassical cousin: the Uzawa-Srinivasan continuous-time version studied in Haque, W. 1970. "Sceptical Notes on Uzawa's 'Optimal Growth in a Two-Sector Model of Capital Accumulation', and a Precise Characterization of the Optimal Path." *The Review of Economic Studies* 37: 377–394. We take Fujio's identification of the marginal rate of transformation ζ of capital goods today into capital goods tomorrow (Fujio, M. 2006. Optimal Transition Dynamics in the Leontief Two-Sector Growth Model. Ph.D. thesis, The Johns Hopkins University), and using a mix of the Bellman-Blackwell methods of dynamic programming and the value-loss approach of Brock-Mitra, pin down the optimal policy for a specific subset of the parameter space. We discern a bifurcation pattern with respect to the discount factor that echoes the results of Benhabib, J., and K. Nishimura. 1979. "The Hopf Bifurcation and the Structure and Stability of Closed Orbits with Heterogeneous Steady States in Multi-Sector Models of Optimal Growth." *Journal of Economic Theory* 21: 421–444 for the neoclassical model, and Khan, M. A., and T. Mitra. 2005. "On Choice of Technique in the Robinson-Solow-Srinivasan Model." *International Journal of Economic Theory* 1: 83–110 for the RSS model. We also study the optimal policy correspondence for a general parameter set.

Keywords: convergence, 2-cycles, golden-rule stock, marginal rate of transformation, optimal policy correspondence, RSL model, RSS model, value function, value-loss

JEL classification: C60, D90, O21

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My first objective is to describe in plain English how the model works, because I think it is an interesting extension of macroeconomics. It seems paradoxical to me that such an important characteristic of the equilibrium path should depend on such a casual property of the technology.[1]

Solow (1962)

I am going to proceed on the assumption that the desirability of a path depends only on the consumption that it gives at every instant of time. Therefore, to steer the economy, the government need only control the allocation of output between consumption and investment. (In a more detailed model, it would have to look after the allocation of investment between the consumption goods industry and the capital-goods industry.)[2]

Solow (1970)

Another outgrowth of the Ramsey work is the development of descriptive models of economic growth where individuals take into consideration explicitly intertemporal considerations. [T]he Ramsey model can be thought of as how a competitive Robinson-Crusoe economy, or a competitive economy with identical individuals living infinitely long, might behave.[3]

Stiglitz and Uzawa (1969)

1   Introduction and antecedent literature

Among his influential contributions to the theory of optimal economic growth, Kazuo Nishimura has always reserved a pride of place for the two-sector model – a model to which Japanese economists from Shinkai, Uzawa, Morishima, Inada, Amano and Takayama downwards have contributed so much, thereby creating an immensely rich tradition in economic theory. It is only fitting that he does so, and through a substantial body of work, rightfully takes his own place in this distinguished line of authorship.[4]

In terms of Nishimura's *oeuvre,* already in his 1979 paper on the application of the Hopf bifurcation to optimal multi-sectoral growth theory, he and Jess Benhabib remind the reader how their basic equations draw on the "duality between the Rybczyski and Stolper-Samuelson effects well known in trade theory."[5] Six years later, in their 1985 analysis of competitive equilibrium cycles, the principal theorems on optimal oscillatory cycles are set in the context of the two-sector model. In what is referred to as a setting where "production is characterized by a two-sector neoclassical nonjoint technology," Nishimura and Benhabib write

> Persistent cycles require a further restriction of the capital-intensity differences between consumption and investment goods. For cycles to be sustained, the oscillations in relative prices must not present intertemporal arbitrage opportunities. Thus possible gains from postponing consumption from periods when the marginal rate of transformation between consumption and investment is high to periods when it is low must not be worth it. Whether this is the case or not depends on the discount rate as well as the slope of the production possibility frontier.

Four years later, in their 1989 work on stochastic oscillatory cycles, Benhahib-Nishimura carry forward their representation of the production-possibility curve in Benhabib-Nishimura (1985, Figure 1) to two "suggestive" figures in a reserved section on diagrammatic exposition.[6]
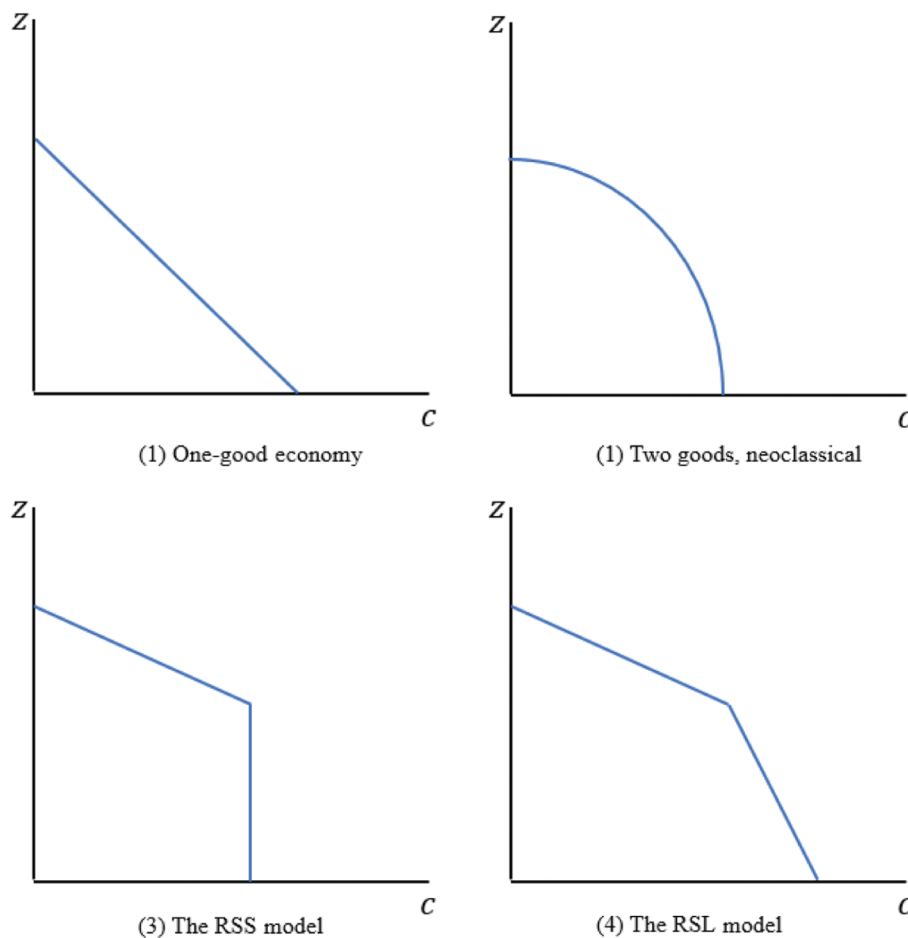

Figure 1: The producion possibility frontier.

However, it is in the late-eighties that the Li-Yorke theorem and topologically chaotic trajectories of perfect-foresight paths enter the picture, and the two-sector model is used in a different exemplary way. Writing in 1994, Nishimura-Yano note that

two types of models are known to have chaotic optimal paths. The examples of Deneckere and Pelikan (1986), Boldrin and Montrucchio (1986), and Boldrin and Deneckere (1990) are based on two-sector models, while that of Majumdar and Mitra (1994) are based on the shape of the utility function.[7]

In what can be seen as the *piece de resistance,* Nishimura and Yano (1995) conclusively

> demonstrate the possibility of ergodically chaotic optimal accumulation in the case [of] a two-sector model with Leontief production functions. [They] construct a condition under which the optimal transition function is unimodal and expansive [and show] that the set of parameter values satisfying that condition is nonempty no matter how weakly the future utilities are discounted.[8]

The direction that this (our) paper takes is, however, best summarized in the words of Boldrin and Deneckere (1990).

> What distinguishes the current paper from the [earlier] research is that we do not construct "artificial" economies that exhibit a pre-chosen dynamics in equilibrium. Rather, we start with a specification of technology and preferences, and *derive* the implied dynamics. While this was also the object of Boldrin's (1989) study, the focus there is on providing criteria for the existence of chaotic paths in abstract two-sector models [see also Deneckere (1988)]. Here we specify two analytically tractable functional forms for the production technologies, and carry out a complete parametric analysis of the resulting dynamics.[9]

This change in viewpoint emphasizing the desirability of a full working-out of a specific model is important and in keeping with May's 1976 emphasis on showing complicated dynamics from simple models. It has been carried forward in the work of Khan-Mitra and Fujio.[10] Indeed, various permutations of the specific versions of the two-sector model are available in the antecedent literature: Leontief-Leontief, Cobb-Douglas-Cobb-Douglas, and Leontief-Cobb-Douglas. In this paper, we name and study the two-sector version of the Robinson-Shinkai-Leontief RSL model.

Leontief's name is pervasive in the work on the two-sector model that we refer to in the paragraphs above, but why Shinkai? why Robinson? As far as the first is concerned, in a 1969 book-chapter titled "A neo-classical passage to growth equilibrium" Morishima (1969) already refers to the "Shinkai-Uzawa finding" and to the "Shinkai-Uzawa capital intensity condition." He writes:

> In the conventional discussion of the stability of growth equilibrium in a two-sector economy, the relative capital intensities of the two industries have served as a kind of litmus paper with which to test whether the Silvery Equilibrium is stable or not. Shinkai observed, for the first time, that the growth equilibrium is stable if and only if the consumption-good industry is more capital-intensive than the capital-good industry."[11]

This is the reason that Fujio in her 2006 dissertation, and in subsequent work,[12] refers to the model as the Leontief-Shinkai (LS) model. However, what seemed to have been missed is Robinson's 1956 contribution. Already, in his 1960 article, Shinkai added a "historical note."

> As the reader will find, our model has much in common with Mrs. Robinson's "simple model" of capital accumulation and Professor Morishima's interpretation of her model, and our equilibrium growth corresponds to her "golden age" (under the hypothesis of no technical progress).[13]

The presentation of the two-sector technology in Robinson's (1956) book can be found in the extensive reviews of that work, the clearest being Barna's (1957), Lancaster's (1960) and Worswick's (1959).[14] However, it is important to bear in mind that we include her name in the naming of both the RSS and RSL models, primarily because of the technological specifications that can be found in her 1956 work, and not simply to use her name in an empty valorization of our enquiry. It is clear that her interests were not in optimal and equilibrium growth, as we take the terms to mean here, but rather in the functioning of a capitalist economy in the macroeconomic short-run and in its leading to a disequilibrium growth process in the long-run in which innovation, gestation lags and expectations play a crucial role.[15]

But this focus on the two-sector technology for the justification of the inclusion of Robinson in the naming of the RSL model blurs the distinction between descriptive and optimal growth. In terms of optimal growth in the smooth neoclassical setting in continuous time, the state of the art seems to be Haque's (1970) analysis.

> In his pioneering essay, Srinivasan (1964) studied the problem of optimal growth in the two-sector model under the assumption that the capital-intensity k_C in the consumption sector is larger at every wage-rental ratio than the capital intensity k_I in the investment sector. He proved the following interesting theorem: If the initial capital-labour ratio is smaller (larger) than a certain critical ratio, then along the

optimal path specialization in investment (consumption) takes place until the critical ratio is attained and then golden age is approached through non-specialization. The optimal path has the special [Srinivasan] property that once specialization is discarded in favour of non-specialization, the former does not re-appear.

It is this factor-intensity assumption, and the Srinivasan property drawing on it, that will constitute the background subtext to the results reported in the sequel.

So much for the framing of the two-sector RSL model in terms of the antecedent literature. The question then arises as to how we contribute substantively and technically to economic growth and to optimal economic dynamics through it. This can best be answered in terms of the plan of the remaining part of the paper. Section 2 presents the basics of the model. It reformulates the model in terms of the Gale-McKenzie reduced form articulated in terms of capital stocks today, and of those tomorrow, and identifies the marginal rate of transformation ζ between them as a crucial sufficiency statistic. This marginal rate had not been identified earlier.[16] Section 3 uses the methods of discounted dynamic programming and offers the preliminary characterizations of the optimal policy correspondence. It is by now well-understood that methods of dynamic programming, while useful for characterization of the optimal trajectory, do not enable the path itself to be fully characterized: one can only guess and verify. Section 4 uses the value-loss approach of Brock-Mitra to pin down the optimal trajectory for the case $\zeta \leq 1$. For the case $\zeta = 1$, we obtain two-period cycles, whereas for the case, $\zeta < 1$, we obtain convergence to the golden-rule stock. This is of consequence in that, in contrast to the antecedent literature, it is the case of $\zeta > 1$ that is responsible for complicated and chaotic dynamics. We collect our findings and conclude the paper in Section 5. The detailed proofs are relegated as supplementary material collected in an online Appendix.

2 The RSL model of choice of technique

Even if we forego short- and intermediate-term macroeconomics, and focus on growth and development, Joan Robinson can be seen to be motivated by the question of the "choice of technique." This was also the primary initial motivation of Khan and Mitra (2005). And so it stands to reason that we do not preclude the RSL model as not being able to say anything regarding this question: it ought to be, as the RSS model, an important special case of multi-sectoral capital theory. In this paper, however, we abstain from these considerations and limit ourselves to the two-sector variant.

2.1 The two-sector RSL model

We consider the two-sector Robinson-Shinkai-Leontief model of optimal economic growth which employs the precise technological specification used in Nishimura and Yano (1995, 1996), and Fujio (2005, 2006, 2008, and 2009) and the literature surveyed in Table 1. There are two production sectors. One unit of consumption good is produced by one unit of labor and a_C units of capital, and b units of investment goods are produced by one unit of labor and a_I units of capital. We only consider the case that the consumption good sector is more capital-intensive than the investment good sector, that is,[17]

$$a_C > a_I. \tag{1}$$

Table 1: Parameter comparison with the antecedent literature.

Model	Consumption Goods		Investment Goods		Depreciation Rate	Marginal Transformation of Capital
	Labor	Capital	Labor	Capital		
RSL	1	a_C	$1/b$	a_I/b	$d \in (0,1)$	$\zeta = b/(a_C - a_I) - (1-d)$
RSS	1	1	a	0	$d \in (0,1)$	$\zeta = 1/a - (1-d)$
Nishimura-Yano	α	1	β/μ	$1/\mu$	$d = 1$	$\alpha\mu/(\beta - \alpha)$
Benhabib-Nishimura	a_{00}	a_{10}	a_{01}	a_{11}	$g \in (0,1)$	$-[a_{01}(a_{11}/a_{01} - a_{10}/a_{00})]^{-1} - (1-g)$

Lancaster	a_{22}	a_{21}	a_{12}	a_{11}	
Worswick I	$1/c$	$1/(cn)$	m	0	$d \in (0,1)$
Worswick II	$1/c$	$1/(cn)$	$1/h$	$1/(hl)$	$d \in (0,1)$

Notes: (1) Benhabib-Nishimura works with a general production function and all the coefficients are those at the steady state; (2) Both Lancaster and Worswick analyze descriptive growth models with a given saving function.

Capital cannot be consumed and depreciates at the rate $d \in (0, 1)$. The case of durable capital nests the RSS model and the Nishimura-Yano model as special or limiting case. This consideration makes the dynamic optimization problem doubly difficult. The amount of capital available at the beginning of next period is equal to the sum of the current production of investment goods and the left-over capital after depreciation. A constant amount of labor, normalized to be unity, is available in each time period $t \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. Thus, in the canonical formulation surveyed in McKenzie (1986, 2005), the collection of production plans (x, x'), the amount x' of capital in the next period (tomorrow) from the amount x of capital available in the current period (today), is given by the *transition possibility set*

$$\Omega = \{(x,x') \in \mathbb{R}_+ \times \mathbb{R}_+ : x' - (1-d)x \geq 0, x' - (1-d)x \leq b\min\{1, x/a_I\}\},$$

where $\mathbb{R}_+$ is the set of non-negative real numbers. For any $(x, x') \in \Omega$, one can consider the amount y of production of consumption goods, leading to a correspondence $\Lambda : \Omega \longrightarrow \mathbb{R}_+$ with

$$\Lambda(x,x') = \{y \in \mathbb{R}_+ : 0 \leq y \leq (1/a_C)(x - (a_I/b)(x' - (1-d)x))$$
$$and \ 0 \leq y \leq 1 - (1/b)(x' - (1-d)x)\}.$$

A felicity function, $w : \mathbb{R}_+ \longrightarrow \mathbb{R}$, is linear and normalized to be equal to the amount of consumption, i.e. $w(y) = y$. Finally, the *reduced form utility function, $u : \Omega \longrightarrow \mathbb{R}_+$,* is defined on Ω such that

$$u(x,x') = \max\{w(y) : y \in \Lambda(x,x')\}.$$

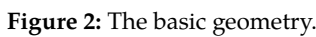We assume that the future welfare is discounted with a discount factor $\rho \in (0, 1)$.[18]

An *economy E* consists of a triple (Ω, u, ρ) and the following concepts apply to it. A *program from x_0* is a sequence $\{x_t, y_t\}$ such that for all $t \in \mathbb{N}$, $(x_t, x_{t+1}) \in \Omega$ and $y_t = \max \Lambda(x_t, x_{t+1})$. A program $\{x_t, y_t\}$ is called *stationary* if for all $t \in \mathbb{N}$, $(x_t, y_t) = (x_{t+1}, y_{t+1})$. For all $0 < \rho < 1$, a program $\{x_t^*, y_t^*\}$ from x_0 is said to be *optimal* if

$$\sum_{t=0}^{\infty} \rho^t [u(x_t, x_{t+1}) - u(x_t^*, x_{t+1}^*)] \leq 0$$

for every program $\{x_t, y_t\}$ from x_0.

2.2 The modified golden rule

Consider the today-tomorrow diagram furnished as Figure 2. Note that the transition possibility set Ω and the reduced-form felicity function $u(\cdot, \cdot)$ have nothing to do with a discount factor. So do the indifference maps. The transition possibility set, Ω, is a set *LVOD* and the indifference maps are kinked lines along with *MV* line; when it is identical to *LVO* its utility level is zero.

Figure 2: The basic geometry.

The modified golden stock $\hat{x}$ is shown as a solution to the following problem;

$$u(\hat{x}, \hat{x}) \geq u(x, x') \text{ for all } (x, x') \in \Omega \text{ such that } x \leq (1 - \rho)\hat{x} + \rho x'. \tag{2}$$

The equation of the line RG is $x' = \rho^{-1}x + \rho^{-1}(\rho - 1)\hat{x}$ specifies a constraint set. It is easily seen that the modified golden-rule stock is exactly equal to the golden-rule stock. It is due to the kinked indifference maps that the modified golden-rule stock is invariant to changes in a discount factor.

In order to guarantee the existence of a nontrivial golden-rule stock, the line OV must be steeper than RG, which is to say,

$$1/\rho < b/a_I + (1 - d) \equiv \theta. \tag{3}$$

We will impose the condition above in what follows.[19] This condition ensures that our technology is *productive*, i.e. there exists a production plan that can produce a positive net amount of investment goods.

The *modified golden rule* is formally defined as a pair $(\hat{x}, \hat{p}) \in \mathbb{R}_+^2$ such that $(\hat{x}, \hat{x}) \in \Omega$ and

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x) \text{ for all } (x, x') \in \Omega. \tag{4}$$

It is easily seen that, in the RSL model, free disposal, bounded paths, and the existence of a stock expansible by the factor ρ^{-1} are satisfied. According to McKenzie (1986), then $\hat{x}$ is the stationary optimal capital stock.

We further introduce a key parameter

$$\zeta \equiv b/(a_C - a_I) - (1 - d)$$

which features prominently in the analysis of the two-sector RSL model. Under the assumption that the consumption good section is more capital intensive, $\zeta > -1$. ζ can be interpreted as the marginal rate of transformation of capital between today and tomorrow under full utilization of capital.[20]

The following proposition formalizes the observations from the simple geometry of the today-tomorrow's diagram.

Proposition 1

There exists a modified golden rule, which is given by

$$(\hat{x}, \hat{p}) = \left(\frac{a_C(\zeta + 1 - d)}{\zeta + 1}, \frac{1}{(a_C - a_I)(1 + \rho\zeta)} \right).$$

2.3 The zero value-loss line and the iso value-loss lines

The *value-loss*[21] $\delta^\rho_{(\hat{p},\hat{x})}(x, x')$ at the golden-rule price system $\hat{p}$ associated with one-period plan (x, x') is defined as the difference between the left-hand side and the right-hand side of the inequality 4

$$\delta^\rho_{(\hat{p},\hat{x})}(x, x') \equiv u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} - u(x, x') - \hat{p}(\rho x' - x) \text{ for all } (x, x') \in \Omega. \tag{5}$$

Lemma 1

There is no value loss when labor and capital are fully utilized.

The line MV in Figure 2 represents a locus of plans with full employment and full capacity utilization. In the triangle VOM, there exits labor not utilized for production of either sector and in the region $LVMD$, there exists excess capacity of capital. As such, the line MV yields zero value-loss at the golden-rule price system $\hat{p}$. This is to say that MV is the von-Neumann facet.

Next, we shall show that the lines parallel to MV are iso-value-loss lines; any two plans on the line have the same amount of value-losses. As an iso-value-loss line moves away from MV line, the associated value-loss increases. With this property, we can see that along the lines OV, MV and MD one-period value-loss is minimized for each given x.

For a plan (x, x') in the triangle MOV which lies below the MV line, we have $(1/a_C)(x - (a_I/b)(x' - (1-d)x)) < 1 - (1/b)(x' - (1 - d)x)$, so the level of utility is written as $u(x, x') = x(b + a_I(1 - d))/(a_C b) - a_I x'/(a_C b)$. Then, the value-loss is:

$$\delta(x, x') = \frac{b - a_I d}{a_C b}\hat{x} + \hat{p}\hat{x}(\rho - 1) - x(b + a_I(1 - d))/(a_C b) + a_I x'/(a_C b) - \hat{p}(\rho x' - x). \tag{6}$$

The marginal change of value-loss with respect to x' is $\partial\delta(x.x')/\partial x' = a_I/(a_C b) - \hat{p}\rho < 0$.

For a plan (x, x') in the region $LVMD$ which lies above the MV line, we have $(1/a_C)(x - (a_I/b)(x' - (1-d)x)) > 1 - (1/b)(x' - (1-d)x)$, so the level of utility is written as $u(x, x') = 1 - x'/b + (1-d)x/b$. Then, the value-loss is:

$$\delta(x, x') = -d\hat{x}/b + \hat{p}\hat{x}(\rho - 1) + x'/b - (1 - d)x/b - \hat{p}(\rho x' - x). \tag{7}$$

The marginal change of value-loss with respect to x' is $\partial\delta(x.x')/\partial x' = 1/b - \hat{p}\rho > 0$. The next result formally establishes this geometric insight, which will be frequently used in our partial characterization of the optimal policy correspondence.

Lemma 2

$a_I/(a_C b) < \hat{p}\rho < 1/b$.

3 The dynamic programming approach

3.1 Preliminaries

In this subsection, we describe the dynamic programming approach with a value function and a policy correspondence to delineate the optimal policy. We define a value function, $V : X \to \mathbb{R}$ by:

$$V(x) = \sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) \tag{8}$$

where $\{x(t), y(t)\}$ is an optimal program starting from $x(0) \in X = (0, \infty)$. The value function is continuous. For each $x \in X$, the Bellman equation:

$$V(x) = \max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\} \tag{9}$$

holds where $\Gamma(x) = \{x' : (x, x') \in \Omega\}$. For each $x \in X$, we denote by $h(x)$ the set of $x' \in \Gamma(x)$ which maximizes $\{u(x, x') + \rho V(x')\}$ among all $x' \in \Gamma(x)$. That is, the optimal policy correspondence is described as,

$$h(x) = \arg \max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\} \tag{10}$$

for each $x \in X$. A program $\{x(t), y(t)\}$ from $x \in X$ is an optimal program from x if and only if it satisfies the equation: $V(x(t)) = u(x(t), x(t+1)) + \rho V(x(t+1))$ for $t \geq 0$.[22]

We now provide some useful properties of the value function which help us partially characterize the optimal dynamics.

Lemma 3

The value function V is (i) concave and (ii) strictly increasing with x.

Lemma 4

The value function, V, satisfies the following properties: i) $V(x) - V(\hat{x}) \leq \hat{p}(x - \hat{x})$ and ii) $V'_+(\hat{x}) \leq \hat{p} \leq V'_-(\hat{x})$.

Since $\hat{x}$ is the stationary optimal stock, we have

$$V(\hat{x}) = \frac{1 - d\hat{x}/b}{1 - \rho} = \frac{b - da_I}{(b + da_C - da_I)(1 - \rho)}.$$

Before solving for the optimal policy correspondence, we provide the following lemma to guide us to partition the set $(0, \infty)$ into sub-regions over which the optimal dynamics tends to behave differently and will be characterized separately.

Lemma 5

If $\zeta > 0$, then $\hat{x}/\theta < a_I < \hat{x} < a_C < \hat{x}/(1-d)$. If $\zeta = 0$, then $\hat{x}/\theta = a_I < \hat{x} < a_C = \hat{x}/(1-d)$. If $\zeta < 0$, then $a_I < \hat{x}/\theta < \hat{x} < \hat{x}/(1-d) < a_C$.

We now turn to characterizing the optimal policy correspondence and the resulting optimal dynamics for different values of ζ.

3.2   A partial characterization

In this subsection, we provide necessary conditions for the optimal policy correspondence for the general parameter space and discuss the convergence property of the system when $\zeta \leq 1$.

3.2.1   The case of positive ζ

In order to derive the optimal policy correspondence, we divide the state space into the following three regions:[23] $(0, \hat{x}/\theta]$, $(\hat{x}/\theta, \hat{x}/(1-d))$, $[\hat{x}/(1-d), \infty)$. We first fully characterize the optimal policy correspondence h, which is single-valued, for the first and third region.

Lemma 6

Let $\zeta > 0$. For $x \in (0, \hat{x}/\theta]$, $h(x) = \{\theta x\}$.

Lemma 7

Let $\zeta > 0$. For $x \in [\hat{x}/(1-d), \infty)$, $h(x) = \{(1-d)x\}$.

The intuition behind Lemma 6 and Lemma 7 is straightforward. If the initial capital stock is sufficiently low, much lower than the golden rule stock, then it is optimal to accumulate capital as quickly as possible and consequently, the economy fully specializes in producing investment goods. On the other hand, if the initial capital stock is sufficiently high, much higher than the golden rule stock, then it is optimal to reduce the capital stock as quickly as possible and as a result, the economy fully specializes in producing consumption goods.

What remains to characterize is the middle region $(\hat{x}/\theta, \hat{x}/(1-d))$. According to Lemma 5, we can further divide the middle region into four subregions: $(\hat{x}/\theta, a_I]$, $(a_I, \hat{x}]$, $(\hat{x}, a_C]$, $(a_C, \hat{x}/(1-d))$.

Proposition 2

Let $\zeta > 0$. For $x \in (\hat{x}/\theta, \hat{x}/(1-d))$, the optimal policy correspondence h satisfies

$$h(x) \subset \begin{cases} [\hat{x}, \theta x] & for \; x \in (\hat{x}/\theta, a_I] \\ [\hat{x}, \zeta(\hat{x} - x) + \hat{x}] & for \; x \in (a_I, \hat{x}] \\ [\zeta(\hat{x} - x) + \hat{x}, \hat{x}] & for \; x \in (\hat{x}, a_C] \\ [(1-d)x, \hat{x}] & for \; x \in (a_C, \hat{x}/(1-d)). \end{cases}$$

Collecting the results above, we obtain the following characterization of the optimal policy correspondence for $\zeta > 0$.

Theorem 1

Let $\zeta > 0$. The optimal policy correspondence h satisfies

$$h(x) \subset G(x) \equiv \begin{cases} \{\theta x\} & for \; x \in (0, \hat{x}/\theta] \\ [\hat{x}, \theta x] & for \; x \in (\hat{x}/\theta, a_I] \\ [\hat{x}, \zeta(\hat{x} - x) + \hat{x}] & for \; x \in (a_I, \hat{x}] \\ [\zeta(\hat{x} - x) + \hat{x}, \hat{x}] & for \; x \in (\hat{x}, a_C] \\ [(1-d)x, \hat{x}] & for \; x \in (a_C, \hat{x}/(1-d)). \\ \{(1-d)x\} & for \; x \in [\hat{x}/(1-d), \infty) \end{cases}$$

The proposition is illustrated in Figure 3. For the middle region, we know that the optimal policy correspondence has to be a selection from the two shaded triangles.
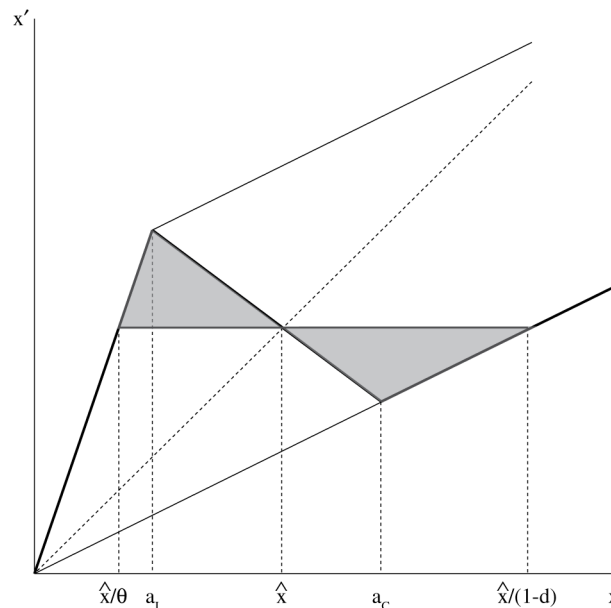

Figure 3: Illustration of Theorem 1.

It should be noticed that the optimal policy correspondence h does not coincide with the optimal policy correspondence of the RSS model as in Khan and Mitra (2007) when $a_I \to 0$. The sharp characterization of the optimal policy for x in $(a_I, \hat{x}]$ in the RSS model hinges on the fact that $a_I = 0$.[24]

Similar to the RSS model, when $0 < \zeta < 1$, the optimal policy leads to global convergence.

Corollary 1

If $\zeta \in (0, 1)$, the stock always converges to the golden rule stock $\hat{x}$.

Moreover, we can show that, if $\zeta = 1$, the stock converges to either the golden rule stock $\hat{x}$ or a two-period cycle. The proof follows from the proof of Corollary 1 with slight modification. If $\theta a_I \leq a_C$, for $x \in [a_I, \hat{x})$, or if $a_I \leq (1-d)a_C$, for $x \in (\hat{x}, a_C]$, $G(x) = [\hat{x}, \zeta(\hat{x} - x) + \hat{x}] = [\hat{x}, 2\hat{x} - x]$ and $G^2(x) = [(1 - \zeta^2)\hat{x} + \zeta^2 x, \hat{x}] = [x, \hat{x}]$, which could give rise to a two-period cycle with periodic points $(\hat{x}, 2\hat{x} - x)$.

To further compare a policy that yields a two-period cycle with a straight-down-to-turnpike policy, consider the case that $\theta a_I \leq a_C$. Let $x \in [a_I, \hat{x}]$. Consider a program under which the stock alternates between x and $2\hat{x} - x$. The utility of this program is given by

$$U_1(x) = \frac{u(x, 2\hat{x} - x) + \rho u(2\hat{x} - x, x)}{1 - \rho^2} = \frac{(2 - d)x}{b(1 + \rho)} - \frac{2(1 - \rho + \rho d)\hat{x}}{b(1 - \rho^2)} + \frac{1}{1 - \rho}.$$

Consider another program under which the stock starts from x and jumps to $\hat{x}$ the next period and stay at the golden rule stock thereafter. The utility of this program is given by

$$U_2(x) = u(x, \hat{x}) + \frac{\rho u(\hat{x}, \hat{x})}{1 - \rho} = \left(\frac{1}{a_C} + \frac{a_I(1 - d)}{a_C b} \right) x - \left(\frac{a_I}{a_C b} + \frac{\rho d}{(1 - \rho)b} \right) \hat{x} + \frac{\rho}{1 - \rho}.$$

By construction, $U_1(\hat{x}) = U_2(\hat{x}) = (1 - d\hat{x}/b)/(1 - \rho)$. Moreover,

$$\frac{\partial U_1(x)}{\partial x} = \frac{2 - d}{b(1 + \rho)} = \frac{b + (2 - d)a_I}{ba_C(1 + \rho)} = \frac{a_I(1 + \theta)}{ba_C(1 + \rho)} < \frac{a_I \theta}{ba_C} = \frac{1}{a_C} + \frac{a_I(1 - d)}{a_C b} = \frac{\partial U_2(x)}{\partial x},$$

where the second equality follows from $\zeta = 1$ and the inequality follows from Condition 3. Since $\partial U_1(x)/\partial x < \partial U_2(x)/\partial x$ and $U_1(\hat{x}) = U_2(\hat{x})$, we must have $U_1(x) > U_2(x)$ for x in $[a_I, \hat{x}]$, which suggests that the two-period policy dominates the straight-down-to-turnpike policy for $\zeta = 1$ and x in $[a_I, \hat{x}]$.

3.2.2 The case of non-positive ζ

For $\zeta \leq 0$, the characterization of high and low initial stocks is the same, while characterization of the middle range becomes simpler.

Proposition 3

Let $\zeta \leq 0$. The optimal policy correspondence h satisfies

$$h(x) \subset \begin{cases} \{\theta x\} & \text{for } x \in (0, a_I] \\ [\zeta(\hat{x} - x) + \hat{x}, \hat{x}] & \text{for } x \in (a_I, \hat{x}] \\ [\hat{x}, \zeta(\hat{x} - x) + \hat{x}] & \text{for } x \in (\hat{x}, a_C) \\ \{(1 - d)x\} & \text{for } x \in [a_C, \infty) \end{cases} \tag{11}$$

The proposition is illustrated in Figure 4, with the two shaded triangles indicating the indeterminate region for the optimal policy. Taking into account the transition possibility set Ω, the optimal policy correspondence for the middle range can further be sharpened as

$$h(x) \subset \begin{cases} [\zeta(\hat{x} - x) + \hat{x}, \min\{\hat{x}, (1 - d)x + b\}] & \text{for } x \in (a_I, \hat{x}] \\ [\max\{\hat{x}, (1 - d)x\}, \zeta(\hat{x} - x) + \hat{x}] & \text{for } x \in (\hat{x}, a_C) \end{cases} \tag{12}$$

Following the similar argument for Corollary 1, the following corollary follows from Proposition 3 and the fact that $\zeta > d - 1 > -1$.

Corollary 2

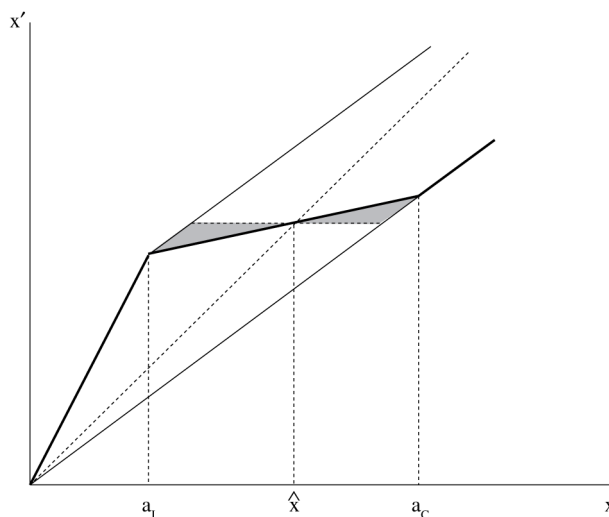*If $\zeta \leq 0$, the stock converges to the golden rule stock $\hat{x}$ monotonically.*

Figure 4: Illustration of Proposition 3.

4   The value-loss approach

4.1   Preliminaries

In this section, based on the value-loss approach as in Khan and Mitra (2006), we sharpen our characterization of the optimal dynamics. In particular, we completely pin down the optimal policy correspondence for $\zeta \leq 1$ and discuss the bifurcation of the optimal policy correspondence with respect to the discount factor. It is worth emphasizing that this type of bifurcation analysis, as in Khan and Mitra (2012), is conceptually different from the bifurcation analysis in the literature of the non-linear dynamics which focuses on primarily how the property of the dynamics of a *given* map changes with the parameter of interest.

We first restate the result described in Section 5 of Khan and Mitra (2006).[25]

Lemma 8
For all $0 < \rho < 1$, a program $\{x_t^*, y_t^*\}$ from x_0 is optimal if and only if

$$\sum_{t=0}^{\infty} \rho^t [\delta^\rho(x_t, x_{t+1}) - \delta^\rho(x_t^*, x_{t+1}^*)] \geq 0$$

for every program $\{x_t, y_t\}$ from x_0.

The lemma says that, in order to find the optimal program that maximizes the total discounted utility, it is equivalent to finding a program that minimizes the total discounted value loss.

4.2   A complete characterization

4.2.1   The case of non-positive ζ

According to Lemma 1, we know that if (x, x') satisfies $x' = \zeta(\hat{x} - x) + \hat{x}$ for $x \in [a_I, a_C]$, there is full utilization of resources and thus, the one-period value loss is zero. Further suggested by Lemma 8, we know that if a program yields zero total value loss, it must be optimal. In light of this insight, we can now characterize the optimal policy *function* for $\zeta \leq 0$.

Proposition 4
Let $\zeta \leq 0$. The optimal policy correspondence is single valued, given by

$$h(x) = \begin{cases} \{\theta x\} & \text{for } x \in (0, a_I] \\ \{\zeta(\hat{x} - x) + \hat{x}\} & \text{for } x \in (a_I, a_C) \\ \{(1-d)x\} & \text{for } x \in [a_C, \infty) \end{cases} \tag{13}$$

The proof of this result is relatively straightforward. We first show that for the initial stock x in the middle range (a_I, a_C), if the next period stock follows $\zeta(\hat{x} - x) + \hat{x}$, then the stock will always stay on the full utilization arm, thus leading to zero total value loss. Any deviation from this policy inevitably leads to positive value loss and therefore is not optimal. For the initial stock that is outside the middle range, we show that the unique optimal policy is to converge to the middle range as fast as possible. Any delay leads to additional value loss.

We turn to sharpening the optimal policy correspondence for $0 < \zeta \leq 1$, which turns out to be much more complicated.

4.2.2   The case of $0 < \zeta \leq 1$

Proposition 5
Let $\zeta \in (0, 1]$. There are three possible cases:
(i) If $a_C \geq a_I \theta$ and $a_C(1 - d) \geq a_I$, then

$$h(x) = \begin{cases} \{\theta x\} & for \ x \in (0, a_I) \\ \{\zeta(\hat{x} - x) + \hat{x}\} & for \ x \in [a_I, a_C] \\ \{(1 - d)x\} & for \ x \in (a_C, \infty) \end{cases} .$$

(ii) If $a_C < a_I\theta$ and $a_C(1 - d) \geq a_I$, then

$$h(x) \subset \begin{cases} \{\theta x\} & for \ x \in (0, a_C/\theta] \\ [a_C, \min\{\theta x, \zeta(\hat{x} - x) + \hat{x}\}] & for \ x \in (a_C/\theta, a_C(\zeta - d)/\zeta) \\ \{\zeta(\hat{x} - x) + \hat{x}\} & for \ x \in [a_C(\zeta - d)/\zeta, a_C] \\ \{(1 - d)x\} & for \ x \in (a_C, \infty) \end{cases} .$$

(iii) If $a_C \geq a_I\theta$ and $a_C(1 - d) < a_I$, then

$$h(x) \subset \begin{cases} \{\theta x\} & for \ x \in (0, a_I) \\ \{\zeta(\hat{x} - x) + \hat{x}\} & for \ x \in [a_I, a_C(1 + (1 - d)/\zeta) - a_I/\zeta] \\ [\max\{\zeta(\hat{x} - x) + \hat{x}, (1 - d)x\}, a_I] & for \ x \in (a_C(1 + (1 - d)/\zeta) - a_I/\zeta, a_I/(1 - d)) \\ \{(1 - d)x\} & for \ x \in [a_I/(1 - d), \infty) \end{cases} .$$

Remark 1

In the RSS model, $a_I = 0$, so $a_C(1 - d) \geq a_I$ always holds. For the special case of our characterization result in the RSS model, see Section 5 of Khan and Mitra (2006). It should be noted that under this extreme assumption that $a_I = 0$, sharper characterization could be obtained for $\zeta > 1$.

Remark 2

Given this refined characterization for $\zeta \leq 1$, the optimal dynamics as in Corollaries Corollary 1 can be easily obtained. In fact, when $\zeta = 1$, the result can be strengthened: the system almost always converges to a two-period cycle.

Case (i) is the simplest case of Proposition 5. The optimal policy is illustrated as the bold line in Figure 5. Similar to the case of $\zeta \leq 0$, the map consists of two arms of specialization and the middle arm of full resource utilization. The proof idea is also similar. Under the condition of Case (i), we show that full utilization of resource for any x starting from the middle range leads to zero value loss. Case (ii), as illustrated in Figure 6, is symmetric to Case (iii), as illustrated in Figure 7. The optimal policy has been pinned down except for the small shaded triangle. The intuition is that if the initial stock starts from the range that gives rise to the triangle, full utilization of resource, despite having a zero value loss for the current period, pushes the next period stock out of the middle range, and therefore, the value loss starting from the next period must be positive. There is a tradeoff between suffering from the value loss this period in order to stay on the middle range and delaying the value loss to the next period, which naturally hinges on the discount factor. The next two propositions fully characterize the optimal policy correspondence for case (ii) and (iii) in Proposition 5, demonstrating an intriguing bifurcation pattern of how optimal policy changes with the discount factor.
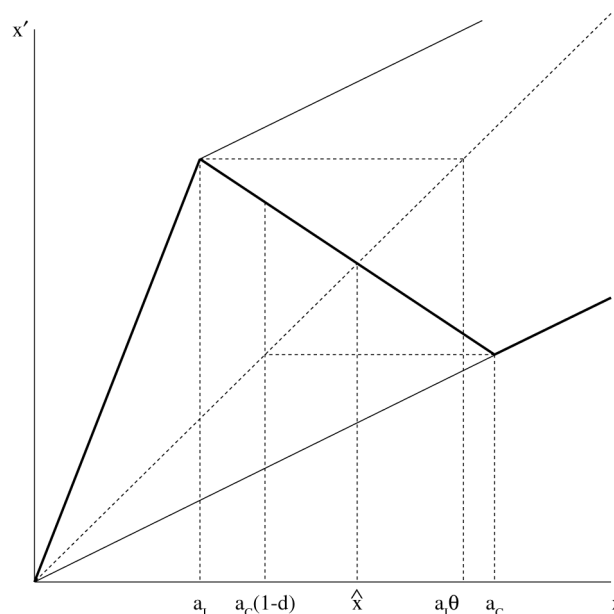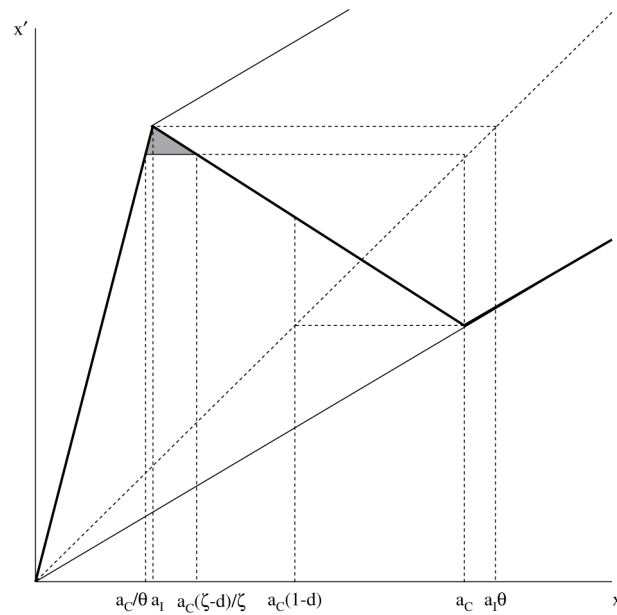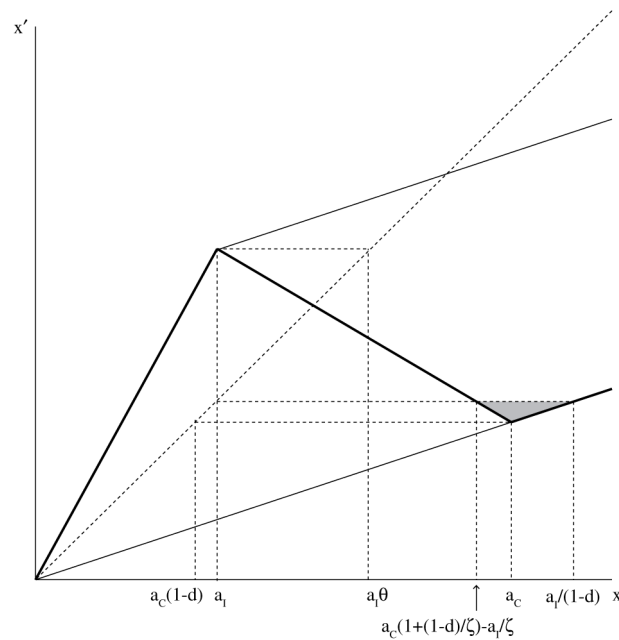
Figure 5: Illustration of Proposition 5: case (i).

Figure 6: Illustration of Proposition 5: case (ii).

Figure 7: Illustration of Proposition 5: case (iii)

Theorem 2

Suppose $\zeta \in (0, 1]$, $a_C < a_I \theta$, and $a_C(1 - d) \geq a_I$. Let t_0 be the smallest integer such that $a_I \theta (1 - d)^{t_0} < a_C$. Let $\bar{\rho}_t$ be the unique positive root to the following equation

$$a_C b (1 - d)^t \rho^{t+1} - (a_C - a_I) a_I \zeta \rho - a_I (a_C - a_I) = 0,$$

for t being the integer satisfying $1 \leq t \leq t_0$. Then $\bar{\rho}_t$ strictly increases with t and $\bar{\rho}_1 > 1/\theta$. Moreover, the optimal policy correspondence for $x \in (a_C/\theta, a_C(\zeta - d)/\zeta)$ depends on ρ, which is given by

$$h(x) = \begin{cases} \{a_C\} & \text{for } \rho < \bar{\rho}_1 \\ [a_C, \min\{\zeta(\hat{x} - x) + \hat{x}, \theta x\}] & \text{for } \rho = \bar{\rho}_1 \\ \min\{\zeta(\hat{x} - x) + \hat{x}, \theta x\} & \text{for } \rho > \bar{\rho}_1 \end{cases},$$

for $t_0 = 1$, and

$$h(x) = \begin{cases} \{a_C\} & \text{for } \rho < \bar{\rho}_1 \\ [\min\{\zeta(\hat{x} - x) + \hat{x}, \theta x, a_C/(1-d)^{t-1}\}, \\ \min\{\zeta(\hat{x} - x) + \hat{x}, \theta x, a_C/(1-d)^t\}] & \text{for } \rho = \bar{\rho}_t, \ t = 1, ..., t_0 \\ \min\{\zeta(\hat{x} - x) + \hat{x}, \theta x, a_C/(1-d)^t\} & \text{for } \bar{\rho}_{t+1} > \rho > \bar{\rho}_t, \ t = 1, ..., t_0 - 1 \\ \min\{\zeta(\hat{x} - x) + \hat{x}, \theta x\} & \text{for } \rho > \bar{\rho}_{t_0} \end{cases},$$

for $t_0 > 1$.

This proposition obtains precisely the optimal policy correspondence for $x \in (a_C/\theta, a_C(\zeta - d)/\zeta)$ of case (ii) in Proposition 5. Generically, the optimal policy correspondence is a function. To understand the characterization, we illustrate an example of $t_0 = 3$ in Figure 8. When $\rho < \bar{\rho}_1$, the optimal policy for the small interval $(a_C/\theta, a_C(\zeta - d)/\zeta)$ is represented by the segment AB. When $\rho = \bar{\rho}_1$, the optimal policy is represented by the trapezoid ADEB. When $\bar{\rho}_1 < \rho < \bar{\rho}_2$, the optimal policy is represented by three segments: AD, DE, and EB. When $\rho = \bar{\rho}_2$, the optimal policy is represented by two segments, AD and EB, and a trapezoid DFGE. When $\bar{\rho}_2 < \rho < \bar{\rho}_3$, the optimal policy is represented by three segments again: AF, FG, and GB. When $\rho = \bar{\rho}_3$, the optimal policy is represented by two segments, AF and GB, and a triangle, FCG. Last, when $\rho > \bar{\rho}_3$, the optimal policy is represented by two segments, AC and CB, which coincides with Case (i) in Proposition 5. Even though it looks complicated, this bifurcation pattern is consistent with our intuition: When the discount factor increases, the planner is more willing to sacrifice today's consumption for tomorrow's gain and the optimal policy gets closer to the case that the economy converges as soon as possible to the full utilization arm.
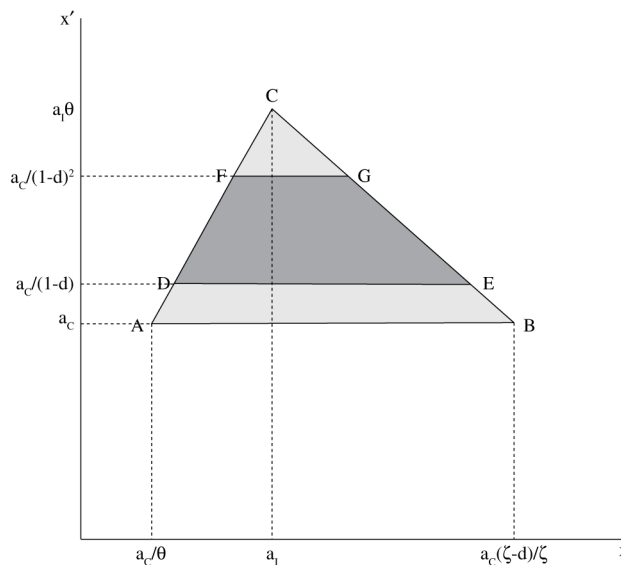

Figure 8: Illustration of Theorem 2.

The main proof idea of this proposition is again based on the value loss approach. We first note that Proposition 5 implies that the economy only suffers from positive value loss for finite periods for any initial stock. Then the key step is to calculate the total discounted value loss for up to $t_0 + 1$ periods for an initial stock in $(a_C/\theta, a_C(\zeta - d)/\zeta)$. The technical complication arises from the fact that the number of periods with positive value loss depends crucially on the policy chosen for the initial stock. We utilize the relationship between the value loss for the finite periods and the discount factor to finally establish the optimality.

Since $\bar{\rho}_1 > 1/\theta$, there always exists a ρ sufficiently close to $1/\theta$ with $\bar{\rho}_1 > \rho > 1/\theta$ such that the optimal policy function has a flat top. Moreover, notice that it is possible to have $\bar{\rho}_1 \geq 1$. For example, consider $a_I = 1$, $a_C = 1.5$, $d = 0.25$, $b = 0.8$. It can be verified that all the conditions in Theorem 2 are satisfied and we have $t_0 = 1$ with $\bar{\rho}_1 \approx 1.018 > 1$. According to Theorem 2, the optimal policy correspondence has a flat top with $h(x) = \{a_C\}$ for any $x \in (a_C/\theta, a_C(\zeta - d)/\zeta)$. For a numerical example that the optimal policy correspondence bifurcates with respect to the discount factor, consider $a_I = 1$, $a_C = 2$, $d = 0.05$, $b = 1.3$. Again, we can verify that all the conditions in Theorem 2 are satisfied and we have $t_0 = 3$ with $\bar{\rho}_1 \approx 0.711$, $\bar{\rho}_2 \approx 0.818$, and $\bar{\rho}_3 \approx 0.875$.

We now refine the characterization for the optimal dynamics for case (iii) in Proposition 5.

Theorem 3

Suppose $\zeta \in (0, 1]$, $a_C \geq a_I\theta$, and $a_C(1 - d) < a_I$. Let t_0 be the smallest integer such that $a_C(1 - d)\theta^{t_0} > a_I$. Let $\tilde{\rho}_t$ be the unique positive root to the following equation

$$b\theta^t\rho^{t+1} - (a_C - a_I)\zeta\rho - (a_C - a_I) = 0,$$

for t being the integer satisfying $1 \leq t \leq t_0$. Then $\tilde{\rho}_t$ strictly decreases with t and $\tilde{\rho}_{t_0} > 1/\theta$. Moreover, the optimal policy correspondence for $x \in (a_C(1 + (1-d)/\zeta) - a_I/\zeta, a_I/(1-d))$ depends on ρ, which is given by

$$h(x) = \begin{cases} \{a_I\} & for\ \rho > \tilde{\rho}_1 \\ [\ \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\}, a_I] & for\ \rho = \tilde{\rho}_1 \\ \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\} & for\ \rho < \tilde{\rho}_1 \end{cases},$$

for $t_0 = 1$, and

$$h(x) = \begin{cases} \{a_I\} & for\ \rho > \tilde{\rho}_1 \\ [\ \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x, a_I/\theta^t\}, & \\ \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x, a_I/\theta^{t-1}\}] & for\ \rho = \tilde{\rho}_t,\ t = 1, ..., t_0 \\ \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x, a_I/\theta^t\} & for\ \tilde{\rho}_{t+1} < \rho < \tilde{\rho}_t,\ t = 1, ..., t_0 - 1 \\ \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\} & for\ \rho < \tilde{\rho}_{t_0} \end{cases},$$

for $t_0 > 1$.

The proof idea is similar to the previous one and the characterization result also looks symmetric. An example for $t_0 = 3$ is provided in the online Appendix.

Since $\tilde{\rho}_{t_0} > 1/\theta$, there always exists a ρ sufficiently close to $1/\theta$ with $\tilde{\rho}_{t_0} > \rho > 1/\theta$ such that the optimal policy has a V-shaped bottom. Again, notice that it is possible to have $\tilde{\rho}_{t_0} \geq 1$. For example, consider $a_I = 1$, $a_C = 1.3$, $d = 0.25$, $b = 0.3$. It can be verified that all the conditions in Theorem 3 are satisfied and we have $t_0 = 1$ with $\tilde{\rho}_1 \approx 1.102 > 1$. According to Theorem 3, the optimal policy correspondence has a V-shaped bottom with $h(x) = \max\{\zeta(\hat{x} - x) + \hat{x}, (1-d)x\}$ for any $x \in (a_C(1 + (1-d)/\zeta) - a_I/\zeta, a_I/(1-d))$. For a numerical example that the optimal policy correspondence bifurcates with respect to the discount factor, consider $a_I = 1$, $a_C = 3$, $d = 0.95$, $b = 2$. Again, we can verify that all the conditions in Theorem 3 are satisfied and we have $t_0 = 3$ with $\tilde{\rho}_1 \approx 0.968$, $\tilde{\rho}_2 \approx 0.740$, and $\tilde{\rho}_3 \approx 0.659$.

Even though we have shown that there is global convergence for the optimal policy when $\zeta < 1$, our analysis above suggests that the optimal policy itself might experience cascade of changes with the discount factor. This stands in contrast with the bifurcation result of the RSS model as in Khan and Mitra (2012). As a special case of our model, they demonstrate that the bifurcation only arises when $\zeta > 1$.

5 Discussion and open questions

In this paper, we provide a partial characterization of the optimal policy correspondence of the RSL model, and based on the value-loss approach, we fully characterize the optimal policy for $\zeta \leq 1$. The characterization results bring out how the optimal policy bifurcates with respect to the discount factor. And so a natural question arises as to the nature of the optimal policy for the case $\zeta > 1$ even for the case of a capital-intensive consumption good sector. We hope to turn to this in a future investigation.

Notes

1 The two sentences are taken from the first two paragraphs of Solow's 1962 11-paragraphed comment on Uzawa's two-sector model of economic growth, reprinted as Chapter 24 in Stiglitz and Uzawa (1969). It is one of the two papers that constitute their Part IV titled *Two-Sector Models* and introduced with the assertion that it, along with Part V titled *Foundations of Optimal Economics,* "represent perhaps the two fastest growing areas of research in growth theory."
2 See Chapter 5 of Solow's 1970 exposition, reprinted in Solow (2000), and commented on in Robinson (1971, p. xii). It testifies to the somewhat conflicted relationship of Solow to the two-sector model: his enthusiastic 1962 reception, its bracketing in 1970, and its complete elimination in the survey (Solow 1999) and the updated (Solow 2000).
3 See the introduction to Part V of the Stiglitz and Uzawa (1969) anthology. We remind the reader that it consists of only four papers: in addition to Ramsey's 1928 and 1958 1958 classics, Malinvaud 1961 analogy between atemporal and intertemporal resource allocation and Solow 1962 comment on the golden rule. All, still eminently worth reading, testify to the profession's understanding of optimal growth theory at that time. The omission of Uzawa 1964 paper on optimal growth is especially interesting from the viewpoint of the results reported in this paper.

4 To be sure, the literature does not have an exclusive Japanese accent: as is well-acknowledged, important contributions have been made by Joan Robinson, James Meade, Ronald Jones, Mordecai Kurz, TN Srinivasan, John Chipman, Joseph Stiglitz and other non-Japanese economists. Once trade theory and political economy is admitted into the picture, the literature simply mushrooms; see Magee (1973), Oswald (1985), and Magee, Brock, and Young (1989), with the bibliographies of the first two references consisting of 162 and 166 items, respectively, and that of the third, running to 49 pages. For a *rapprochement* of trade and growth theory, see Fujio and Khan (2006); and for a synthesis stemming from the Oniki-Uzawa model, Chen, Nishimura, and Shimomura (2004).

5 See Chapter 9 reprinted in Benhabib (1992), and Figure 1 below. Also see Chapter 13 of Benhabib (1992), and the recurrence of these theorems in Benhabib-Nishimura-Venditti (2002, Section 6.2). Even though our emphasis in this work is on optimal growth in the two-sector model, we refer the reader to Burmeister-Dobell (1970, Chapters 4-5) and (Dixit, 1976, Chapter 5) for equilibrium growth. For the impact of these classical theorems on modern political economy, see subject index in Magee, Brock, and Young (1989).

6 See Figures 13.1 and 13.2 in Section 2 in Benhabib and Nishimura (1989) reprinted in Benhabib (1992, pp. 291–293).

7 See Footnote 3 in of Nishimura and Yano (1994) reprinted in Majumdar-Mitra-Nishimura (2000, p. 149). To save on references, we send the reader to the bibliography of the anthology.

8 For other examples concerning both topological and ergodic chaos based on the two-sector model with Leontief production functions, (what we are calling the two-sector RSL model here), see Chapters 4, 8 and 9 in Majumdar, Mitra, and Nishimura (2000).

9 See the fourth paragraph of Boldrin and Deneckere (1990) reprinted in Benhabib (1992).

10 For the first, see Khan and Mitra (2005, 2013) and their references; and for the second, see Fujio (2005, 2006, 2008, and 2009).

11 See Morishima (1969, p. 45), and his book more generally for the definitions of his terms; also Robinson (1962, pp. 51-63), and his Lectures 1 and 2 in Morishima (1965). In Morishima (1969), Morishima continues, "Uzawa replaced Shinkai's production functions of the Leontief type by the neo-classical ones which allow continuous substitution between labour and capital, to find that the relative capital-intensity criterion is a sufficient condition for stability but no longer a necessary condition — though Furuno later saw that the Shinkai-Uzawa finding should be subject to a proviso that the introduction of a production lag narrows the stability region."

12 See the references to Fujio in Footnote 10.

13 There are only five references in Shinkai (1960): one of these is to Robinson's 1956 book, and the other to a 1958 unpublished note of Morishima.

14 See pp. 46-47 in Robinson (1962, 1971), and in addition to Robinson herself, see Worswick (1963), Hamberg (1963), and Little (1957).

15 See, for example, her evaluation of Meade's book on two-sector growth in the Conclusion in Robinson (1971), the footnote to Uzawa in p. 132, more generally, her reminiscences in Robinson (1978), and the Harcourt-Kerr introduction to the second edition of Robinson (1956). Also see Robinson (1970) and her exchange with Stiglitz referred to in Khan and Mitra (2005).

16 See Table 1. This sufficiency statistic is due to Fujio (2006) for the case of undiscounted optimal growth, and builds on earlier work of Khan and Mitra (2005) for the RSS model. Also see McKenzie (1986) and his references for the Gale-McKenzie reduced form.

17 The same assumption is imposed in Nishimura and Yano (1995) and Fujio (2006). We pursue the case for $a_C \leq a_I$ in a different paper.

18 For a comprehensive treatment of the undiscounted case ($\rho = 1$), see Fujio (2006).

19 The RSS model can be viewed as a special case of the RSL model with $a_C = 1$, $a_I = 0$, and $b = 1/a$. Since $a_I = 0$, Condition 3 is always satisfied in the RSS model. Also see Condition 2.10 in Nishimura and Yano (1995) for the same condition with $d = 1$.

20 Note that the counterpart of ζ in the RSS model is ξ as in, for example, Khan and Mitra (2012). It is assumed that $\xi > 1$. For discussion about $\xi \leq 1$, see Khan and Mitra (2006).

21 We shall abbreviate $\delta^\rho_{(\hat{p}, \hat{x})}(x, x')$ by $\delta^\rho(x, x')$.

22 We refer interested readers to chapter 4 of Le Van and Dana (2003) for a general treatment and rigorous proofs of the value function being well-defined, the existence of optimal program, and the value function being the unique continuous solution to the Bellman equation.

23 By definition, $\theta > (1 - d)$, so the three intervals are pairwise disjoint.

24 See the proof of their Lemma 2. For a sharp characterization of the optimal dynamics of the RSS model, also see Khan and Mitra (2012) and especially their Figure 1.

25 The proof of the result see their online Appendix. The main idea is to establish the following equation

$$\sum_{t=0}^{\infty} \rho^t \left[u(x_t^*, x_{t+1}^*) - u(x_t, x_{t+1}) \right] = \sum_{t=0}^{\infty} \rho^t \left[\delta^\rho(x_t, x_{t+1}) - \delta^\rho(x_t^*, x_{t+1}^*) \right],$$

for $\{x_t, y_t\}$ and $\{x_t^*, y_t^*\}$ that start from the same initial stock.

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